

A generalized eigenvalue algorithm for tridiagonal matrix pencils based on a nonautonomous discrete integrable system

Kazuki Maeda^{a,*}, Satoshi Tsujimoto^a

^aDepartment of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan

Abstract

A generalized eigenvalue algorithm for tridiagonal matrix pencils is presented. The algorithm appears as the time evolution equation of a nonautonomous discrete integrable system associated with a polynomial sequence which has some orthogonality on the support set of the zeros of the characteristic polynomial for a tridiagonal matrix pencil. The convergence of the algorithm is discussed by using the solution to the initial value problem for the corresponding discrete integrable system.

Keywords: generalized eigenvalue problem, nonautonomous discrete integrable system, R_{II} chain, dqds algorithm, orthogonal polynomials

2010 MSC: 37K10, 37K40, 42C05, 65F15

1. Introduction

Applications of discrete integrable systems to numerical algorithms are important and fascinating topics. Since the end of the twentieth century, a number of relationships between classical numerical algorithms and integrable systems have been studied (see the review papers [1–3]). On this basis, new algorithms based on discrete integrable systems have been developed: (i) singular value algorithms for bidiagonal matrices based on the discrete Lotka–Volterra equation [4, 5], (ii) Padé approximation algorithms based on the discrete relativistic Toda lattice [6] and the discrete Schur flow [7], (iii) eigenvalue algorithms for band matrices based on the discrete hungry Lotka–Volterra equation [8] and the nonautonomous discrete hungry Toda lattice [9], and (iv) algorithms for computing D-optimal designs based on the nonautonomous discrete Toda (nd-Toda) lattice [10] and the discrete modified KdV equation [11].

In this paper, we focus on a nonautonomous discrete integrable system called the R_{II} chain [12], which is associated with the generalized eigenvalue problem for tridiagonal matrix pencils [13]. The relationship between the finite R_{II} chain and the generalized eigenvalue problem can be understood to be an analogue of the connection between the *finite nd-Toda lattice* and the eigenvalue problem for tridiagonal matrices. In numerical analysis, the time evolution equation of the finite nd-Toda lattice is called the *dqds (differential quotient difference with shifts) algorithm* [14], which is well known as a fast and accurate iterative algorithm for computing eigenvalues or singular values. Therefore, it is worth to consider the application of the finite R_{II} chain to algorithms for computing generalized eigenvalues. The purpose of this paper is to construct a generalized eigenvalue algorithm based on the finite R_{II} chain and to prove the convergence of the algorithm. Further improvements and comparisons with traditional methods will be studied in subsequent papers.

The nd-Toda lattice on a semi-infinite lattice or a non-periodic finite lattice has a *Hankel determinant solution*. In the background, there are *monic orthogonal polynomials*, which give rise to this solution; monic orthogonal polynomials have a determinant expression that relates to the Hankel determinant, and spectral

*Corresponding author

Email addresses: kmaeda@amp.i.kyoto-u.ac.jp (Kazuki Maeda), tsujimoto@i.kyoto-u.ac.jp (Satoshi Tsujimoto)

transformations for monic orthogonal polynomials give the Lax pair of the nd-Toda lattice [15, 16]. Especially, for the finite lattice case, we can easily solve the initial value problem for the nd-Toda lattice with the Gauss quadrature formula for *monic finite orthogonal polynomials*. This special property of the discrete integrable system allows us to analyze the behaviour of the system in detail and tells us how parameters should be chosen to accelerate the convergence of the dqds algorithm. We will give a review of this theory in Section 2.

The theory above will be extended to the R_{II} chain in Section 3. The three-term recurrence relation that monic orthogonal polynomials satisfy arises from a tridiagonal matrix. In a similar way, a tridiagonal matrix pencil defines a monic polynomial sequence. This polynomial sequence, called monic R_{II} polynomials [17], possesses similar properties to monic orthogonal polynomials and their spectral transformations yield the monic type R_{II} chain. A determinant expression of the monic R_{II} polynomials gives a Hankel determinant solution and, in particular for the finite lattice case, a convergence theorem of the monic R_{II} chain is shown under an assumption. This theorem enables us to design a generalized eigenvalue algorithm.

The dqds algorithm is a subtraction-free algorithm, i.e., the recurrence equations of the dqds algorithm do not contain subtraction operations except origin shifts (see Subsection 2.3). The subtraction-free form is numerically effective to avoid the loss of significant digits. In addition, there is another application of the subtraction-free form: ultradiscretization [18] or tropicalization [19]; e.g., the ultradiscretization of the finite nd-Toda lattice in a subtraction-free form gives a time evolution equation of the box-ball system with a carrier [20]. In Section 4, for the monic type R_{II} chain, we will present its subtraction-free form, which contains no subtractions except origin shifts under some conditions. It is considered that this form makes the computation of the proposed algorithm more accurate. At the end of the paper, numerical examples will be presented to confirm that the proposed algorithm computes the generalized eigenvalues of given tridiagonal matrix pencils fast and accurately.

2. Monic orthogonal polynomials, nd-Toda lattice, and dqds algorithm

First, we will review the connection between the theory of orthogonal polynomials and the nd-Toda lattice.

2.1. Infinite dimensional case

Let us consider a tridiagonal semi-infinite matrix of the form

$$B^{(t)} = \begin{pmatrix} u_0^{(t)} & 1 & & \\ w_1^{(t)} & u_1^{(t)} & 1 & \\ & w_2^{(t)} & u_2^{(t)} & 1 \\ & & w_3^{(t)} & \ddots & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \quad u_n^{(t)} \in \mathbb{C}, \quad w_n^{(t)} \in \mathbb{C} - \{0\},$$

where $t \in \mathbb{N}$ is the discrete time, whose evolution will be introduced later. Let I_n denote the identity matrix of order n and $B_n^{(t)}$ the n -th order leading principal submatrix of $B^{(t)}$. We now introduce a polynomial sequence $\{\phi_n^{(t)}(x)\}_{n=0}^\infty$:

$$\phi_0^{(t)}(x) := 1, \quad \phi_n^{(t)}(x) := \det(xI_n - B_n^{(t)}), \quad n = 1, 2, 3, \dots$$

By definition, $\phi_n^{(t)}(x)$ is a monic polynomial of degree n . The Laplace expansion for $\det(xI_{n+1} - B_{n+1}^{(t)})$ with respect to the last row yields the three-term recurrence relation

$$\phi_{n+1}^{(t)}(x) = (x - u_n^{(t)})\phi_n^{(t)}(x) - w_n^{(t)}\phi_{n-1}^{(t)}(x), \quad n = 0, 1, 2, \dots, \quad (2.1)$$

where we set $w_0^{(t)} := 0$ and $\phi_{-1}^{(t)}(x) := 0$. It is well known that the three-term recurrence relation of the form (2.1) gives the following classical theorem.

Theorem 2.1 (Favard's Theorem [21, Chapter I, Section 4]). *For the polynomials $\{\phi_n^{(t)}(x)\}_{n=0}^\infty$ satisfying the three-term recurrence relation (2.1) and any nonzero constant $h_0^{(t)}$, there exists a unique linear functional $\mathcal{L}^{(t)}$ defined on the space of all polynomials such that the orthogonality relation*

$$\mathcal{L}^{(t)}[x^m \phi_n^{(t)}(x)] = h_n^{(t)} \delta_{m,n}, \quad n = 0, 1, 2, \dots, \quad m = 0, 1, \dots, n, \quad (2.2)$$

holds, where

$$h_n^{(t)} = h_0^{(t)} w_1^{(t)} w_2^{(t)} \dots w_n^{(t)}, \quad n = 1, 2, 3, \dots,$$

and $\delta_{m,n}$ is Kronecker delta.

From the relation (2.2), we readily obtain the relation

$$\mathcal{L}^{(t)}[\phi_m^{(t)}(x) \phi_n^{(t)}(x)] = h_n^{(t)} \delta_{m,n}, \quad m, n = 0, 1, 2, \dots$$

Therefore, the polynomials $\{\phi_n^{(t)}(x)\}_{n=0}^\infty$ are the monic orthogonal polynomials with respect to $\mathcal{L}^{(t)}$.

Let us define the *moment* of order m by

$$\mu_m^{(t)} := \mathcal{L}^{(t)}[x^m], \quad m = 0, 1, 2, \dots,$$

and its *Hankel determinant* of order n by

$$\tau_0^{(t)} := 1, \quad \tau_n^{(t)} := |\mu_{i+j}^{(t)}|_{i,j=0}^{n-1} = \begin{vmatrix} \mu_0^{(t)} & \mu_1^{(t)} & \dots & \mu_{n-1}^{(t)} \\ \mu_1^{(t)} & \mu_2^{(t)} & \dots & \mu_n^{(t)} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}^{(t)} & \mu_n^{(t)} & \dots & \mu_{2n-2}^{(t)} \end{vmatrix}, \quad n = 1, 2, 3, \dots$$

Since the monic orthogonal polynomials with respect to $\mathcal{L}^{(t)}$ are uniquely determined, we then find the determinant expression of the polynomial $\phi_n^{(t)}(x)$:

$$\phi_n^{(t)}(x) = \frac{1}{\tau_n^{(t)}} \begin{vmatrix} \mu_0^{(t)} & \mu_1^{(t)} & \dots & \mu_{n-1}^{(t)} & \mu_n^{(t)} \\ \mu_1^{(t)} & \mu_2^{(t)} & \dots & \mu_n^{(t)} & \mu_{n+1}^{(t)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1}^{(t)} & \mu_n^{(t)} & \dots & \mu_{2n-2}^{(t)} & \mu_{2n-1}^{(t)} \\ 1 & x & \dots & x^{n-1} & x^n \end{vmatrix}, \quad n = 0, 1, 2, \dots \quad (2.3)$$

Next, we introduce the discrete time evolution into the monic orthogonal polynomials by the following transformation from $\{\phi_n^{(t)}(x)\}_{n=0}^\infty$ to $\{\phi_n^{(t+1)}(x)\}_{n=0}^\infty$:

$$(x - s^{(t)}) \phi_n^{(t+1)}(x) = \phi_{n+1}^{(t)}(x) + q_n^{(t)} \phi_n^{(t)}(x), \quad n = 0, 1, 2, \dots, \quad (2.4)$$

where

$$q_n^{(t)} := -\frac{\phi_{n+1}^{(t)}(s^{(t)})}{\phi_n^{(t)}(s^{(t)})}, \quad n = 0, 1, 2, \dots, \quad (2.5)$$

and $s^{(t)}$ is a parameter that is not a zero of $\phi_n^{(t)}(x)$ for all $n = 0, 1, 2, \dots$. Suppose that $\{\phi_n^{(t)}(x)\}_{n=0}^\infty$ are the monic orthogonal polynomials with respect to $\mathcal{L}^{(t)}$ and define a new linear functional $\mathcal{L}^{(t+1)}$ by

$$\mathcal{L}^{(t+1)}[P(x)] := \mathcal{L}^{(t)}[(x - s^{(t)})P(x)] \quad (2.6)$$

for all polynomials $P(x)$. Then, it is easily verified that $\{\phi_n^{(t+1)}(x)\}_{n=0}^\infty$ are monic orthogonal polynomials with respect to $\mathcal{L}^{(t+1)}$ again. Since it is shown that the monic orthogonal polynomials satisfy the three-term recurrence relation of the form (2.1), another relation

$$\phi_n^{(t)}(x) = \phi_n^{(t+1)}(x) + e_n^{(t)} \phi_{n-1}^{(t+1)}(x), \quad n = 0, 1, 2, \dots, \quad (2.7)$$

is derived for consistency. The variable $e_n^{(t)}$ satisfies the compatibility condition

$$u_n^{(t+1)} = q_n^{(t+1)} + e_n^{(t+1)} + s^{(t+1)} = q_n^{(t)} + e_{n+1}^{(t)} + s^{(t)}, \quad (2.8a)$$

$$w_n^{(t+1)} = q_{n-1}^{(t+1)} e_n^{(t+1)} = q_n^{(t)} e_n^{(t)} \quad (2.8b)$$

with the boundary condition

$$e_0^{(t)} = 0 \quad \text{for all } t \geq 0. \quad (2.8c)$$

The transformations (2.4) and (2.7) are called the *Christoffel transformation* and the *Geronimus transformation*, respectively [22]. The discrete dynamical system (2.8) is the semi-infinite nd-Toda lattice.

We have seen above the derivation of the nd-Toda lattice from the theory of orthogonal polynomials. Using this connection, we can give an explicit solution to the semi-infinite nd-Toda lattice (2.8); the solution is written in terms of the moments of the monic orthogonal polynomials. The time evolution of the linear functional (2.6) leads to

$$\mu_m^{(t+1)} = \mu_{m+1}^{(t)} - s^{(t)} \mu_m^{(t)}. \quad (2.9)$$

By applying this relation to the determinant expression of the monic orthogonal polynomials (2.3), the definition of the variable (2.5) yields

$$q_n^{(t)} = \frac{\tau_n^{(t)} \tau_{n+1}^{(t+1)}}{\tau_{n+1}^{(t)} \tau_n^{(t+1)}}. \quad (2.10)$$

Further, applying the orthogonality relation (2.2) to equation (2.7), we obtain

$$e_n^{(t)} = \frac{\mathcal{L}^{(t)}[x^n \phi_n^{(t)}(x)]}{\mathcal{L}^{(t+1)}[x^{n-1} \phi_{n-1}^{(t+1)}(x)]} = \frac{h_n^{(t)}}{h_{n-1}^{(t+1)}} = \frac{\tau_{n+1}^{(t)} \tau_{n-1}^{(t+1)}}{\tau_n^{(t)} \tau_n^{(t+1)}}. \quad (2.11)$$

If the moments $\mu_m^{(t)}$, the elements of the Hankel determinant $\tau_n^{(t)}$, are arbitrary functions satisfying the relation (2.9), then these (2.10) and (2.11) give particular solutions to the semi-infinite nd-Toda lattice (2.8). For instance,

$$\mu_m^{(t)} = \int_{\Omega} x^m \prod_{j=0}^{t-1} (x - s^{(j)}) \omega(x) dx$$

satisfies the relation (2.9), where Ω is an interval of the real line and $\omega(x)$ is a weight function on Ω . If the integral of the right-hand side has a finite value for all $m \in \mathbb{N}$, then this moment gives a solution.

2.2. Finite dimensional case

In what follows, we shall reduce the size of the tridiagonal matrix $B^{(t)}$ to finite N :

$$B^{(t)} := \begin{pmatrix} u_0^{(t)} & 1 & & & \\ w_1^{(t)} & u_1^{(t)} & 1 & & \\ & w_2^{(t)} & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & w_{N-1}^{(t)} & u_{N-1}^{(t)} \end{pmatrix}.$$

The matrix $B^{(t)}$ determines a system of *monic finite orthogonal polynomials*. The corresponding nd-Toda lattice is also reduced to the case of the non-periodic finite lattice of size N :

$$q_n^{(t+1)} + e_n^{(t+1)} + s^{(t+1)} = q_n^{(t)} + e_{n+1}^{(t)} + s^{(t)}, \quad (2.12a)$$

$$q_{n-1}^{(t+1)} e_n^{(t+1)} = q_n^{(t)} e_n^{(t)}, \quad (2.12b)$$

$$e_0^{(t)} = e_N^{(t)} = 0 \quad \text{for all } t \geq 0. \quad (2.12c)$$

We can solve the initial value problem for the finite nd-Toda lattice (2.12) through the theory of finite orthogonal polynomials.

The monic finite orthogonal polynomials $\{\phi_n^{(t)}(x)\}_{n=0}^N$ are defined in the same way as the infinite dimensional case: $\phi_n^{(t)}(x) := \det(xI_n - B_n^{(t)})$. It should be remarked that $\phi_N^{(t)}(x)$ is the characteristic polynomial of $B^{(t)}$. For the polynomials $\{\phi_n^{(t)}(x)\}_{n=0}^N$ and any nonzero constant $h_0^{(t)}$, there exists a unique linear functional $\mathcal{L}^{(t)}$ such that the orthogonality relation

$$\mathcal{L}^{(t)}[x^m \phi_n^{(t)}(x)] = h_n^{(t)} \delta_{m,n}, \quad n = 0, 1, \dots, N-1, \quad m = 0, 1, \dots, n, \quad (2.13a)$$

and the terminating condition

$$\mathcal{L}^{(t)}[x^m \phi_N^{(t)}(x)] = 0, \quad m = 0, 1, 2, \dots, \quad (2.13b)$$

hold.

Let x_0, x_1, \dots, x_{N-1} denote the zeros of the characteristic polynomial $\phi_N^{(t)}(x)$, i.e.,

$$\phi_N^{(t)}(x) = \prod_{i=0}^{N-1} (x - x_i). \quad (2.14)$$

If for simplicity we assume that these zeros are all simple, the linear functional $\mathcal{L}^{(t)}$ is concretely given by the *Gauss quadrature formula*.

Theorem 2.2 (Gauss quadrature formula [21, Chapter I, Section 6]). *Let x_0, x_1, \dots, x_{N-1} be the simple zeros of the characteristic polynomial $\phi_N^{(t)}(x)$. For the linear functional $\mathcal{L}^{(t)}$ of the monic finite orthogonal polynomials $\{\phi_n^{(t)}(x)\}_{n=0}^N$, there exist some constants $c_0^{(t)}, c_1^{(t)}, \dots, c_{N-1}^{(t)}$ such that*

$$\mathcal{L}^{(t)}[P(x)] = \sum_{i=0}^{N-1} c_i^{(t)} P(x_i) \quad (2.15)$$

holds for all polynomials $P(x)$. Further, if $w_1^{(t)}, w_2^{(t)}, \dots, w_{N-1}^{(t)}$ are all real and positive, then $c_0^{(t)}, c_1^{(t)}, \dots, c_{N-1}^{(t)}$ are also all real and positive.

The formula (2.15) means that the monic finite orthogonal polynomials $\{\phi_n^{(t)}(x)\}_{n=0}^N$ with the terminating condition (2.13b) are orthogonal on the support set of the zeros x_0, x_1, \dots, x_{N-1} of the characteristic polynomial $\phi_N^{(t)}(x)$.

The constants $c_0^{(t)}, c_1^{(t)}, \dots, c_{N-1}^{(t)}$ are calculated as

$$c_i^{(t)} = \frac{h_{N-1}^{(t)}}{\phi_{N-1}^{(t)}(x_i) \phi_N^{(t)'}(x_i)}, \quad i = 0, 1, \dots, N-1, \quad (2.16)$$

where $\phi_N^{(t)}(x)$ is the derivative of $\phi_N^{(t)}(x)$. This formula is verified as follows. Due to the Gauss quadrature formula (2.15), the moment is given by

$$\mu_m^{(t)} = \mathcal{L}^{(t)}[x^m] = \sum_{i=0}^{N-1} c_i^{(t)} x_i^m, \quad m = 0, 1, 2, \dots$$

This yields the relation

$$\mu_{m+1}^{(t)} - x_j \mu_m^{(t)} = \sum_{\substack{i=0 \\ i \neq j}}^{N-1} c_i^{(t)} (x_i - x_j) x_i^m, \quad j = 0, 1, \dots, N-1, \quad m = 0, 1, 2, \dots$$

This relation and the determinant expression of the monic orthogonal polynomials (2.3) lead to

$$\phi_{N-1}^{(t)}(x_j) = \frac{1}{\tau_{N-1}^{(t)}} \prod_{\substack{i=0 \\ i \neq j}}^{N-1} c_i^{(t)} (x_j - x_i) \prod_{\substack{0 \leq \nu_0 < \nu_1 \leq N-1 \\ \nu_0 \neq j, \nu_1 \neq j}} (x_{\nu_1} - x_{\nu_0})^2, \quad j = 0, 1, \dots, N-1.$$

A similar calculation yields

$$\tau_N^{(t)} = \prod_{i=0}^{N-1} c_i^{(t)} \prod_{0 \leq \nu_0 < \nu_1 \leq N-1} (x_{\nu_1} - x_{\nu_0})^2.$$

Further, we have

$$\phi_N^{(t)}(x_j) = \prod_{\substack{i=0 \\ i \neq j}}^{N-1} (x_j - x_i), \quad j = 0, 1, \dots, N-1,$$

$$h_{N-1}^{(t)} = \mathcal{L}^{(t)}[x^{N-1} \phi_{N-1}^{(t)}(x)] = \frac{\tau_N^{(t)}}{\tau_{N-1}^{(t)}}.$$

These equations lead to the formula (2.16).

The spectral transformations (2.4) and (2.7) also work for the finite dimensional case except the Christoffel transformation for $n = N$:

$$\phi_N^{(t+1)}(x) = \phi_N^{(t)}(x), \quad (2.17)$$

which is consistent with the Geronimus transformation for $n = N$. Equation (2.17) means that the characteristic polynomial $\phi_N^{(t)}(x)$ of the tridiagonal matrix $B^{(t)}$ is invariant under the time evolution. In other words, the time evolution of the monic finite orthogonal polynomials does not change the eigenvalues of the tridiagonal matrix $B^{(t)}$. Since the time evolution of the moment is given by (2.9), we have the following expression for the moment:

$$\mu_m^{(t)} = \sum_{i=0}^{N-1} \left(c_i^{(0)} x_i^m \prod_{j=0}^{t-1} (x_i - s^{(j)}) \right), \quad (2.18)$$

where we define $t = 0$ as the initial time. Substituting this expression of the moment (2.18) into the elements of the Hankel determinant $\tau_n^{(t)}$ and applying the Binet–Cauchy formula and the Vandermonde determinant formula, we obtain the expanded form of $\tau_n^{(t)}$:

$$\tau_n^{(t)} = \sum_{0 \leq r_0 < r_1 < \dots < r_{n-1} \leq N-1} \left(\prod_{i=0}^{n-1} \left(c_{r_i}^{(0)} \prod_{j=0}^{t-1} (x_{r_i} - s^{(j)}) \right) \prod_{0 \leq \nu_0 < \nu_1 \leq n-1} (x_{r_{\nu_1}} - x_{r_{\nu_0}})^2 \right). \quad (2.19)$$

Hence, we can conclude that the solution to the initial value problem for the finite nd-Toda lattice (2.12) is given by

$$q_n^{(t)} = \frac{\tau_n^{(t)} \tau_{n+1}^{(t+1)}}{\tau_{n+1}^{(t)} \tau_n^{(t+1)}}, \quad e_n^{(t)} = \frac{\tau_{n+1}^{(t)} \tau_{n-1}^{(t+1)}}{\tau_n^{(t)} \tau_n^{(t+1)}}, \quad (2.20)$$

with the expanded form of $\tau_n^{(t)}$ (2.19) and the expression of $c_n^{(0)}$ (2.16).

In the rest of this subsection, we will reformulate the matrix forms of the finite nd-Toda lattice. Let $L^{(t)}$ and $R^{(t)}$ be bidiagonal matrices of order N :

$$L^{(t)} = \begin{pmatrix} 1 & & & & \\ e_1^{(t)} & 1 & & & \\ & e_2^{(t)} & \ddots & & \\ & & \ddots & \ddots & \\ & & & e_{N-1}^{(t)} & 1 \end{pmatrix}, \quad R^{(t)} = \begin{pmatrix} q_0^{(t)} & 1 & & & \\ & q_1^{(t)} & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & q_{N-1}^{(t)} \end{pmatrix},$$

and let $\boldsymbol{\phi}^{(t)}(x)$ and $\boldsymbol{\phi}_N^{(t)}(x)$ be the vectors of order N :

$$\boldsymbol{\phi}^{(t)}(x) = \begin{pmatrix} \phi_0^{(t)}(x) \\ \phi_1^{(t)}(x) \\ \vdots \\ \phi_{N-1}^{(t)}(x) \end{pmatrix}, \quad \boldsymbol{\phi}_N^{(t)}(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \phi_N^{(t)}(x) \end{pmatrix}.$$

Then, the three-term recurrence relation (2.1) and the spectral transformations (2.4) and (2.7) are written as

$$B^{(t)} \boldsymbol{\phi}^{(t)}(x) + \boldsymbol{\phi}_N^{(t)}(x) = x \boldsymbol{\phi}^{(t)}(x), \quad (2.21a)$$

$$(x - s^{(t)}) \boldsymbol{\phi}^{(t+1)}(x) = R^{(t)} \boldsymbol{\phi}^{(t)}(x) + \boldsymbol{\phi}_N^{(t)}(x), \quad (2.21b)$$

$$\boldsymbol{\phi}^{(t)}(x) = L^{(t)} \boldsymbol{\phi}^{(t+1)}(x). \quad (2.21c)$$

Note that, from (2.14) and (2.21a),

$$B^{(t)} \boldsymbol{\phi}^{(t)}(x_i) = x_i \boldsymbol{\phi}^{(t)}(x_i), \quad i = 0, 1, \dots, N-1,$$

holds. This corresponds to the fact that the zeros of the characteristic polynomials $\phi_N^{(t)}(x)$ are the eigenvalues of $B^{(t)}$. Moreover, it indicates that $\boldsymbol{\phi}^{(t)}(x_i)$ is the eigenvector corresponding to the eigenvalue x_i . From (2.21), we have

$$\begin{aligned} x \boldsymbol{\phi}^{(t+1)}(x) &= B^{(t+1)} \boldsymbol{\phi}^{(t+1)}(x) + \boldsymbol{\phi}_N^{(t+1)}(x) \\ &= (L^{(t+1)} R^{(t+1)} + s^{(t+1)} I_N) \boldsymbol{\phi}^{(t+1)}(x) + \boldsymbol{\phi}_N^{(t+1)}(x) \\ &= (R^{(t)} L^{(t)} + s^{(t)} I_N) \boldsymbol{\phi}^{(t+1)}(x) + \boldsymbol{\phi}_N^{(t+1)}(x). \end{aligned}$$

This yields the matrix form of the finite nd-Toda lattice (2.12):

$$B^{(t+1)} = L^{(t+1)} R^{(t+1)} + s^{(t+1)} I_N = R^{(t)} L^{(t)} + s^{(t)} I_N.$$

Since $L^{(t)}$ is always regular, it is shown that the tridiagonal matrices $B^{(t)}$ and $B^{(t+1)}$ are similar:

$$B^{(t+1)} = R^{(t)} L^{(t)} + s^{(t)} I_N = \left(L^{(t)} \right)^{-1} \left(L^{(t)} R^{(t)} + s^{(t)} I_N \right) L^{(t)} = \left(L^{(t)} \right)^{-1} B^{(t)} L^{(t)}.$$

Therefore, the eigenvalues of $B^{(t)}$ are conserved under the time evolution. This corresponds to the fact (2.17) that the characteristic polynomial $\phi_N^{(t)}(x)$ is invariant under the time evolution. From the result, we can see that the spectral transformations (2.21b) and (2.21c) correspond to the LU decomposition of the tridiagonal matrix $B^{(t)}$ with the shift $s^{(t)}$.

2.3. The dqds algorithm

For the finite nd-Toda lattice (2.12), let us introduce an auxiliary variable

$$d_n^{(t+1)} := q_n^{(t+1)} - e_{n+1}^{(t)}, \quad n = 0, 1, \dots, N-1. \quad (2.22)$$

Then, equations (2.12) are rewritten as

$$d_0^{(t+1)} = q_0^{(t)} - (s^{(t+1)} - s^{(t)}), \quad (2.23a)$$

$$d_n^{(t+1)} = d_{n-1}^{(t+1)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} - (s^{(t+1)} - s^{(t)}), \quad n = 1, 2, \dots, N-1, \quad (2.23b)$$

$$q_n^{(t+1)} = e_{n+1}^{(t)} + d_n^{(t+1)}, \quad n = 0, 1, \dots, N-1, \quad (2.23c)$$

$$e_n^{(t+1)} = e_n^{(t)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}}, \quad n = 1, 2, \dots, N-1, \quad (2.23d)$$

$$e_0^{(t)} = e_N^{(t)} = 0 \quad \text{for all } t \geq 0. \quad (2.23e)$$

These recurrence equations are called the *dqds algorithm*.

The spectral transformations (2.4) and (2.7) yields

$$\phi_n^{(t+1)}(x) = \frac{\phi_{n+1}^{(t-1)}(x) + d_n^{(t)} \phi_n^{(t)}(x)}{x - s^{(t)}}, \quad n = 0, 1, \dots, N-1.$$

Hence, we obtain

$$d_n^{(t)} = -\frac{\phi_{n+1}^{(t-1)}(s^{(t)})}{\phi_n^{(t)}(s^{(t)})} = \frac{\tau_n^{(t)} \sigma_{n+1}^{(t)}}{\tau_{n+1}^{(t-1)} \tau_n^{(t+1)}}, \quad (2.24)$$

with

$$\sigma_n^{(t)} := |\mu_{i+j+1}^{(t-1)} - s^{(t)} \mu_{i+j}^{(t-1)}|_{0 \leq i, j \leq n-1}.$$

By a calculation similar to the derivation of the expanded form (2.19), we obtain the expanded form of $\sigma_n^{(t)}$

$$\sigma_n^{(t)} = \sum_{0 \leq r_0 < r_1 < \dots < r_{n-1} \leq N-1} \left(\prod_{i=0}^{n-1} \left(c_{r_i}^{(0)} (x_{r_i} - s^{(t)}) \prod_{j=0}^{t-2} (x_{r_i} - s^{(j)}) \right) \prod_{0 \leq \nu_0 < \nu_1 \leq n-1} (x_{r_{\nu_1}} - x_{r_{\nu_0}})^2 \right). \quad (2.25)$$

We henceforth assume that, for the elements of the initial tridiagonal matrix $B^{(0)}$, the following conditions are satisfied: $u_0^{(0)}, u_1^{(0)}, \dots, u_{N-1}^{(0)}$ are all real and $w_1^{(0)}, w_2^{(0)}, \dots, w_{N-1}^{(0)}$ are all real and positive. Then, the tridiagonal matrix $B^{(0)}$ is similar to a real symmetric tridiagonal matrix. The eigenvalues x_0, x_1, \dots, x_{N-1} of $B^{(0)}$ are thus all real and simple. In addition, the constants $c_0^{(0)}, c_1^{(0)}, \dots, c_{N-1}^{(0)}$ are all real and positive by Theorem 2.2. Accordingly, the solution (2.20) and (2.24) with the expanded forms (2.19) and (2.25) gives the next theorem.

Theorem 2.3. Suppose that $u_0^{(0)}, u_1^{(0)}, \dots, u_{N-1}^{(0)}$ are all real, and $w_1^{(0)}, w_2^{(0)}, \dots, w_{N-1}^{(0)}$ are all real and positive. Choose the parameter $s^{(t)}$ as

$$s^{(t)} < \min\{x_0, x_1, \dots, x_{N-1}\} \quad \text{for all } t \geq 0. \quad (2.26)$$

Then, the variables $q_n^{(t)}, e_n^{(t)}$ and $d_n^{(t)}$ of the dqds algorithm (2.23) are positive for all n and $t \geq 0$.

This theorem guarantees that, under the assumption, the dqds algorithm does not contain subtraction operations except the parameter terms $-(s^{(t+1)} - s^{(t)})$ in equations (2.23a) and (2.23b). Namely, equations (2.23) are the subtraction-free form of the finite nd-Toda lattice (2.12). It is known that this form improves the accuracy of the numerical computation.

The asymptotic analysis of the dqds algorithm proves convergence of the algorithm and provides a method for accelerating the convergence. The solution derived in Subsection 2.2 is a key tool for the analysis. Arrange the eigenvalues x_0, x_1, \dots, x_{N-1} of the initial tridiagonal matrix $B^{(0)}$ in descending order: $x_0 > x_1 > \dots > x_{N-1}$. If the parameter $s^{(t)}$ chosen as (2.26), namely $s^{(t)} < x_{N-1}$, then the inequality $x_0 - s^{(t)} > x_1 - s^{(t)} > \dots > x_{N-1} - s^{(t)} > 0$ holds for all $t \geq 0$. Under this assumption, by the solution (2.20) with the expanded form (2.19), we obtain the asymptotic behaviour for $t \gg 0$:

$$q_n^{(t)} = x_n - s^{(t)} + O\left(\max\left\{\frac{\prod_{j=0}^t(x_n - s^{(j)})}{\prod_{j=0}^{t-1}(x_{n-1} - s^{(j)})}, \frac{\prod_{j=0}^t(x_{n+1} - s^{(j)})}{\prod_{j=0}^{t-1}(x_n - s^{(j)})}\right\}\right), \quad e_n^{(t)} = O\left(\frac{\prod_{j=0}^{t-1}(x_n - s^{(j)})}{\prod_{j=0}^t(x_{n-1} - s^{(j)})}\right).$$

This shows that $q_n^{(t)}$ and $e_n^{(t)}$ converge to $x_n - s^{(t)}$ and 0 as $t \rightarrow +\infty$, respectively. Hence, it is shown that the dqds algorithm (2.23) with appropriate parameters $s^{(t)}$ computes the eigenvalues of a given real symmetric tridiagonal matrix. It is clear that the convergence speed depends on $\frac{x_n - s^{(t)}}{x_{n-1} - s^{(t)}}$. Therefore, we should choose the parameter $s^{(t)} < x_{N-1}$ as close as possible to the minimum eigenvalue x_{N-1} for fast computation. The acceleration parameter $s^{(t)}$ is called the *origin shift*.

3. R_{II} polynomials, R_{II} chain, and generalized eigenvalue algorithm

We shall extend the discussion in Section 2 for tridiagonal matrix pencils and its associated nonautonomous discrete integrable system.

3.1. Infinite dimensional case

Let us consider two tridiagonal semi-infinite matrices in the following forms:

$$A^{(t)} = \begin{pmatrix} v_0^{(t)} & \kappa_t & & \\ \lambda_1 w_1^{(t)} & v_1^{(t)} & \kappa_{t+1} & \\ & \lambda_2 w_2^{(t)} & v_2^{(t)} & \kappa_{t+2} \\ & & \lambda_3 w_3^{(t)} & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \quad v_n^{(t)}, \kappa_{t+n}, \lambda_n \in \mathbb{C}, \quad w_n^{(t)} \in \mathbb{C} - \{0\},$$

$$B^{(t)} = \begin{pmatrix} u_0^{(t)} & 1 & & \\ w_1^{(t)} & u_1^{(t)} & 1 & \\ & w_2^{(t)} & u_2^{(t)} & 1 \\ & & w_3^{(t)} & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \quad u_n^{(t)} \in \mathbb{C}.$$

Let $A_n^{(t)}$ and $B_n^{(t)}$ denote the n -th order leading principal submatrices of $A^{(t)}$ and $B^{(t)}$, respectively. We now define a polynomial sequence $\{\varphi_n^{(t)}(x)\}_{n=0}^\infty$ by

$$\varphi_0^{(t)}(x) := 1, \quad \varphi_n^{(t)}(x) := \det(xB_n^{(t)} - A_n^{(t)}), \quad n = 1, 2, 3, \dots$$

The polynomial $\varphi_n^{(t)}(x)$ is a monic polynomial of degree n . In the same manner as in the case of monic orthogonal polynomials in Section 2, we obtain the three-term recurrence relation

$$\varphi_{n+1}^{(t)}(x) = (u_n^{(t)}x - v_n^{(t)})\varphi_n^{(t)}(x) - w_n^{(t)}(x - \kappa_{t+n-1})(x - \lambda_n)\varphi_{n-1}^{(t)}(x), \quad n = 0, 1, 2, \dots, \quad (3.1)$$

where we set $w_0^{(t)} := 0$ and $\varphi_{-1}^{(t)}(x) := 0$. We will assume in what follows that all the parameters κ_{t+k} and λ_k , $k = 0, 1, 2, \dots$, are not zeros of the polynomial $\varphi_n^{(t)}(x)$ for all $n \in \mathbb{N}$. The polynomials $\{\varphi_n^{(t)}(x)\}_{n=0}^\infty$ are called the R_{II} polynomials with respect to $\mathcal{L}^{(t)}$, introduced by Ismail and Masson [17].

We introduce the notations

$$K_k^{(t)}(x) := \prod_{j=0}^{k-1} (x - \kappa_{t+j}), \quad K_k(x) := K_k^{(0)}(x) = \prod_{j=0}^{k-1} (x - \kappa_j), \quad L_l(x) := \prod_{j=1}^l (x - \lambda_j),$$

and $\mathcal{D}(\mathcal{L}^{(t)})$ a linear space spanned by the rational functions $\frac{x^m}{K_k^{(t)}(x)L_l(x)}$, $k, l = 0, 1, 2, \dots$; $m = 0, 1, \dots, k + l$.

The following Favard type theorem is proved.

Theorem 3.1 (Favard type theorem for the R_{II} polynomials [17]). *For the R_{II} polynomials $\{\varphi_n^{(t)}(x)\}_{n=0}^\infty$ and any nonzero constants $h_0^{(t)}$ and $h_1^{(t)}$, which satisfy $h_0^{(t)} \neq h_1^{(t)}$, there exists a unique linear functional defined on $\mathcal{D}(\mathcal{L}^{(t)})$ such that the orthogonality relation*

$$\mathcal{L}^{(t)} \left[\frac{x^m \varphi_n^{(t)}(x)}{K_n^{(t)}(x)L_n(x)} \right] = h_n^{(t)} \delta_{m,n}, \quad n = 0, 1, 2, \dots, \quad m = 0, 1, 2, \dots, n,$$

holds, where $h_n^{(t)}$, $n = 2, 3, \dots$, are some nonzero constants.

In the rest of this paper, we consider the *monic* R_{II} polynomials, i.e., the case where $u_n^{(t)} = 1 + w_n^{(t)}$ holds for all $n = 0, 1, 2, \dots$. For general tridiagonal semi-infinite matrices of the form $B^{(t)}$, if $\det(B_n^{(t)}) \neq 0$ holds for all $n = 1, 2, 3, \dots$, then $\left\{ \frac{\varphi_n^{(t)}(x)}{\det(B_n^{(t)})} \right\}_{n=0}^\infty$ are the monic R_{II} polynomials. Therefore, the following argument is valid for such matrices.

The moment of the R_{II} linear functional $\mathcal{L}^{(t)}$ is introduced by

$$\mu_m^{k,l,t} := \mathcal{L}^{(t)} \left[\frac{x^m}{K_k^{(t)}(x)L_l(x)} \right], \quad k, l = 0, 1, 2, \dots, \quad m = 0, 1, \dots, k + l, \quad (3.2)$$

and its Hankel determinant by

$$\tau_0^{k,l,t} := 1, \quad \tau_n^{k,l,t} := |\mu_{i+j}^{k,l,t}|_{0 \leq i,j \leq n-1} = \begin{vmatrix} \mu_0^{k,l,t} & \mu_1^{k,l,t} & \dots & \mu_{n-1}^{k,l,t} \\ \mu_1^{k,l,t} & \mu_2^{k,l,t} & \dots & \mu_n^{k,l,t} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}^{k,l,t} & \mu_n^{k,l,t} & \dots & \mu_{2n-2}^{k,l,t} \end{vmatrix}, \quad n = 1, 2, 3, \dots$$

Then, the determinant expression of the monic R_{II} polynomials $\{\varphi_n^{(t)}(x)\}_{n=0}^\infty$ is presented:

$$\varphi_n^{(t)}(x) = \frac{1}{\tau_n^{n,n,t}} \begin{vmatrix} \mu_0^{n,n,t} & \mu_1^{n,n,t} & \dots & \mu_{n-1}^{n,n,t} & \mu_n^{n,n,t} \\ \mu_1^{n,n,t} & \mu_2^{n,n,t} & \dots & \mu_n^{n,n,t} & \mu_{n+1}^{n,n,t} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1}^{n,n,t} & \mu_n^{n,n,t} & \dots & \mu_{2n-2}^{n,n,t} & \mu_{2n-1}^{n,n,t} \\ 1 & x & \dots & x^{n-1} & x^n \end{vmatrix}, \quad n = 0, 1, 2, \dots \quad (3.3)$$

The discrete time evolution for the monic R_{II} polynomials is introduced by an analogue of the spectral transformations for monic orthogonal polynomials:

$$(x - s^{(t)})(1 + q_n^{(t)})\varphi_n^{(t+1)}(x) = \varphi_{n+1}^{(t)}(x) + q_n^{(t)}(x - \kappa_{t+n})\varphi_n^{(t)}(x), \quad (3.4a)$$

$$(1 + e_n^{(t)})\varphi_n^{(t)}(x) = \varphi_n^{(t+1)}(x) + e_n^{(t)}(x - \lambda_n)\varphi_{n-1}^{(t+1)}(x) \quad (3.4b)$$

for $n = 0, 1, 2, \dots$, where

$$q_n^{(t)} := -(s^{(t)} - \kappa_{t+n}) \frac{\varphi_{n+1}^{(t)}(s^{(t)})}{\varphi_n^{(t)}(s^{(t)})}, \quad n = 0, 1, 2, \dots, \quad (3.5)$$

and $e_n^{(t)}$ is the variable determined by the compatibility condition:

$$\begin{aligned} 1 + w_n^{(t+1)} &= -q_n^{(t+1)} - e_n^{(t+1)} \frac{1 + q_n^{(t+1)}}{1 + q_{n-1}^{(t+1)}} + (1 + q_n^{(t+1)})(1 + e_n^{(t+1)}) \\ &= -q_n^{(t)} \frac{1 + e_{n+1}^{(t)}}{1 + e_n^{(t)}} - e_{n+1}^{(t)} + (1 + q_n^{(t)})(1 + e_{n+1}^{(t)}), \end{aligned} \quad (3.6a)$$

$$\begin{aligned} v_n^{(t+1)} &= -\kappa_{t+n+1} q_n^{(t+1)} - \lambda_n e_n^{(t+1)} \frac{1 + q_n^{(t+1)}}{1 + q_{n-1}^{(t+1)}} + s^{(t+1)}(1 + q_n^{(t+1)})(1 + e_n^{(t+1)}) \\ &= -\kappa_{t+n} q_n^{(t)} \frac{1 + e_{n+1}^{(t)}}{1 + e_n^{(t)}} - \lambda_{n+1} e_{n+1}^{(t)} + s^{(t)}(1 + q_n^{(t)})(1 + e_{n+1}^{(t)}), \end{aligned} \quad (3.6b)$$

$$w_n^{(t+1)} = q_{n-1}^{(t+1)} e_n^{(t+1)} \frac{1 + q_n^{(t+1)}}{1 + q_{n-1}^{(t+1)}} = q_n^{(t)} e_n^{(t)} \frac{1 + e_{n+1}^{(t)}}{1 + e_n^{(t)}} \quad (3.6c)$$

with the boundary condition

$$e_0^{(t)} = 0 \quad \text{for all } t \geq 0. \quad (3.6d)$$

This is the semi-infinite *monic type R_{II} chain*. Note that, since (3.6a) and (3.6c) are identical, there are the two independent equations that determine the time evolution of the two variables $q_n^{(t)}$ and $e_n^{(t)}$. It is readily verified that if $\{\varphi_n^{(t)}(x)\}_{n=0}^\infty$ are the monic R_{II} polynomials with respect to $\mathcal{L}^{(t)}$, then polynomials $\{\varphi_n^{(t+1)}(x)\}_{n=0}^\infty$ defined by the spectral transformation (3.4a) are again the monic R_{II} polynomials, where the corresponding R_{II} linear functional is defined by

$$\mathcal{L}^{(t+1)}[R(x)] := \mathcal{L}^{(t)} \left[\frac{x - s^{(t)}}{x - \kappa_t} R(x) \right] \quad (3.7)$$

for all $R(x) \in \mathcal{D}(\mathcal{L}^{(t+1)})$.

Let us derive a solution to the monic type R_{II} chain. By the definition of the moment (3.2) and the time evolution of the linear functional (3.7), we obtain the relations

$$\mu_m^{k,l,t} = \mu_{m+1}^{k+1,l,t} - \kappa_{t+k} \mu_m^{k+1,l,t} = \mu_{m+1}^{k,l+1,t} - \lambda_{l+1} \mu_m^{k,l+1,t}, \quad (3.8a)$$

$$\mu_m^{k,l,t+1} = \mu_{m+1}^{k+1,l,t} - s^{(t)} \mu_m^{k+1,l,t}. \quad (3.8b)$$

The relation (3.8b), the determinant expression of the monic R_{II} polynomials (3.3), and the definition of the variable (3.5) lead to

$$q_n^{(t)} = (s^{(t)} - \kappa_{t+n})^{-1} \frac{\tau_n^{n,n,t} \tau_{n+1}^{n,n+1,t+1}}{\tau_n^{n-1,n,t+1} \tau_{n+1}^{n+1,n+1,t}}. \quad (3.9)$$

Next, the relation (3.8a) and the spectral transformation (3.4a) yield

$$1 + q_n^{(t)} = (\kappa_{t+n} - s^{(t)})^{-1} \frac{\varphi_{n+1}^{(t)}(\kappa_{t+n})}{\varphi_n^{(t+1)}(\kappa_{t+n})} = (s^{(t)} - \kappa_{t+n})^{-1} \frac{\tau_{n+1}^{n,n+1,t} \tau_n^{n,n,t+1}}{\tau_{n+1}^{n+1,n+1,t} \tau_n^{n-1,n,t+1}}.$$

Further, the Jacobi identity for determinants [23, Section 2.6] proves the bilinear equation

$$\tau_n^{k-1,l-1,t} \tau_n^{k,l,t} - \tau_n^{k-1,l,t} \tau_n^{k,l-1,t} - \tau_{n-1}^{k-1,l-1,t} \tau_{n+1}^{k,l,t} = 0.$$

By using this bilinear equation and the three-term recurrence relation (3.1), we obtain

$$w_n^{(t)} = \frac{\mathcal{L}^{(t)} \left[\frac{x^{n+1} \phi_{n+1}^{(t)}(x)}{K_{n+1}^{(t)}(x) L_{n+1}(x)} - \frac{x^n \phi_n^{(t)}(x)}{K_n^{(t)}(x) L_n(x)} \right]}{\mathcal{L}^{(t)} \left[\frac{x^n \phi_n^{(t)}(x)}{K_n^{(t)}(x) L_n(x)} - \frac{x^{n-1} \phi_{n-1}^{(t)}(x)}{K_{n-1}^{(t)}(x) L_{n-1}(x)} \right]} = \frac{\tau_{n-1}^{n-1,n-1,t} \tau_{n+1}^{n,n+1,t} \tau_{n+1}^{n+1,n,t}}{\tau_n^{n-1,n,t} \tau_n^{n,n-1,t} \tau_{n+1}^{n+1,n+1,t}}.$$

Hence, from equation (3.6c) and these formulae, we find a solution

$$e_n^{(t)} = \frac{w_n^{(t)}}{q_{n-1}^{(t)}} \frac{1 + q_{n-1}^{(t)}}{1 + q_n^{(t)}} = (s^{(t)} - \kappa_{t+n}) \frac{\tau_{n-1}^{n-1,n-1,t+1} \tau_{n+1}^{n+1,n,t}}{\tau_n^{n,n-1,t} \tau_n^{n,n,t+1}}. \quad (3.10)$$

If the moments $\mu_m^{k,l,t}$ are arbitrary functions satisfying the relations (3.8), e.g.,

$$\mu_m^{k,l,t} = \int_{\Omega} \frac{x^m \prod_{j=0}^{t-1} (x - s^{(j)})}{K_{t+n}(x) L_n(x)} \omega(x) dx,$$

then (3.9) and (3.10) give a solution to the monic type R_{II} chain (3.6) expressed by the Hankel determinant $\tau_n^{k,l,t}$.

The reason why the Hankel determinant appears can be explained from the point of view of the discrete two-dimensional Toda hierarchy [24]. Note that there is another determinant expression of the R_{II} polynomials and a solution to the R_{II} chain: the Casorati-type determinant solution [25, 26].

3.2. Finite dimensional case

In this subsection, we will derive the solution to the initial value problem and the convergence theorem for the monic type finite R_{II} chain.

Let us start with a pair of tridiagonal matrices of order N :

$$A^{(t)} = \begin{pmatrix} v_0^{(t)} & \kappa_t & & & \\ \lambda_1 w_1^{(t)} & v_1^{(t)} & \kappa_{t+1} & & \\ & \lambda_2 w_2^{(t)} & \ddots & \ddots & \\ & & \ddots & \ddots & \kappa_{t+N-2} \\ & & & \lambda_{N-1} w_{N-1}^{(t)} & v_{N-1}^{(t)} \end{pmatrix}, \quad B^{(t)} = \begin{pmatrix} 1 & 1 & & & \\ w_1^{(t)} & 1 + w_1^{(t)} & 1 & & \\ & w_2^{(t)} & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & w_{N-1}^{(t)} & 1 + w_{N-1}^{(t)} \end{pmatrix}. \quad (3.11)$$

The corresponding monic type finite R_{II} chain is

$$\begin{aligned} \kappa_{t+n+1} q_n^{(t+1)} + \lambda_n e_n^{(t+1)} \frac{1 + q_n^{(t+1)}}{1 + q_{n-1}^{(t+1)}} - s^{(t+1)} (1 + q_n^{(t+1)}) (1 + e_n^{(t+1)}) \\ = \kappa_{t+n} q_n^{(t)} \frac{1 + e_{n+1}^{(t)}}{1 + e_n^{(t)}} + \lambda_{n+1} e_{n+1}^{(t)} - s^{(t)} (1 + q_n^{(t)}) (1 + e_{n+1}^{(t)}), \end{aligned} \quad (3.12a)$$

$$q_{n-1}^{(t+1)} e_n^{(t+1)} \frac{1 + q_n^{(t+1)}}{1 + q_{n-1}^{(t+1)}} = q_n^{(t)} e_n^{(t)} \frac{1 + e_{n+1}^{(t)}}{1 + e_n^{(t)}}, \quad (3.12b)$$

$$e_0^{(t)} = e_N^{(t)} = 0 \quad \text{for all } t \geq 0. \quad (3.12c)$$

To derive the solution to the initial value problem for the monic type finite R_{II} chain (3.12), we consider the monic finite R_{II} polynomials $\{\varphi_n^{(t)}(x)\}_{n=0}^\infty$ defined by $\varphi_n^{(t)}(x) := \det(xB_n^{(t)} - A_n^{(t)})$. We should remark that $\varphi_N^{(t)}(x)$ is the characteristic polynomial of the tridiagonal matrix pencil $(A^{(t)}, B^{(t)})$; the zeros of the polynomial $\varphi_N^{(t)}(x)$ are the generalized eigenvalues of the matrix pencil $(A^{(t)}, B^{(t)})$, i.e., the solutions of the equation

$$A^{(t)}\Phi = xB^{(t)}\Phi, \quad x \in \mathbb{C}, \quad \Phi \in \mathbb{C}^N - \{0\}.$$

Let $\mathcal{D}(\mathcal{L}^{(t)})$ be a linear space spanned by the rational functions $\frac{x^m}{K_N^{(t)}(x)L_N(x)}$, $m = 0, 1, 2, \dots$. For the monic finite R_{II} polynomials $\{\varphi_n^{(t)}(x)\}_{n=0}^N$ and any nonzero constant $H^{(t)}$, there exists a unique linear functional defined on $\mathcal{D}(\mathcal{L}^{(t)})$ such that the orthogonality relation

$$\mathcal{L}^{(t)} \left[\frac{x^m \varphi_n^{(t)}(x)}{K_n^{(t)}(x)L_n(x)} \right] = h_n^{(t)} \delta_{m,n}, \quad n = 0, 1, \dots, N-1, \quad m = 0, 1, \dots, n, \quad (3.13a)$$

and the terminating condition

$$\mathcal{L}^{(t)} \left[\frac{x^m \varphi_N^{(t)}(x)}{K_k^{(t)}(x)L_l(x)} \right] = 0, \quad k, l = 0, 1, \dots, N, \quad m = 0, 1, 2, \dots, \quad (3.13b)$$

hold, where the constants $h_0^{(t)}, h_1^{(t)}, \dots, h_{N-1}^{(t)}$ are given by solving the following linear equation

$$\begin{pmatrix} -1 & 1 & & & & \\ w_1^{(t)} & -(1+w_1^{(t)}) & 1 & & & \\ & w_2^{(t)} & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & w_{N-1}^{(t)} & -(1+w_{N-1}^{(t)}) & \end{pmatrix} \begin{pmatrix} h_0^{(t)} \\ h_1^{(t)} \\ \vdots \\ h_{N-1}^{(t)} \end{pmatrix} = \begin{pmatrix} -H^{(t)} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

i.e.,

$$\begin{aligned} h_0^{(t)} &= H^{(t)}(1 + w_1^{(t)} + w_1^{(t)}w_2^{(t)} + \dots + w_1^{(t)}w_2^{(t)} \dots w_{N-1}^{(t)}), \\ h_1^{(t)} &= H^{(t)}(w_1^{(t)} + w_1^{(t)}w_2^{(t)} + \dots + w_1^{(t)}w_2^{(t)} \dots w_{N-1}^{(t)}), \\ &\vdots \\ h_{N-1}^{(t)} &= H^{(t)}w_1^{(t)}w_2^{(t)} \dots w_{N-1}^{(t)}. \end{aligned}$$

Note that, for the infinite dimensional case, there are two degrees of freedom: the choice of the two constants $h_0^{(t)}$ and $h_1^{(t)}$ (see Theorem 3.1). For the finite dimensional case, however, there is only one degree of freedom: the choice of the constant $H^{(t)}$. The cause of this is the terminating condition (3.13b).

To derive a realization of $\mathcal{L}^{(t)}$, we give a quadrature formula for the R_{II} linear functional. Suppose that all the zeros x_0, x_1, \dots, x_{N-1} of the characteristic polynomial $\varphi_N^{(t)}(x)$ are simple.

Theorem 3.2 (The quadrature formula for the R_{II} linear functional). *Let x_0, x_1, \dots, x_{N-1} be the simple zeros of the characteristic polynomial $\varphi_N^{(t)}(x)$. For the linear functional $\mathcal{L}^{(t)}$ of the monic finite R_{II} polynomials $\{\varphi_n^{(t)}(x)\}_{n=0}^N$, there exist some constants $c_0^{(t)}, c_1^{(t)}, \dots, c_{N-1}^{(t)}$ such that*

$$\mathcal{L}^{(t)}[R(x)] = \sum_{i=0}^{N-1} c_i^{(t)} R(x_i) \quad (3.14)$$

holds for all $R(x) \in \mathcal{D}(\mathcal{L}^{(t)})$.

Proof. This proof is an analogue of the proof to the Gauss quadrature formula (Theorem 2.2). For the given rational function $R(x)$, consider the following interpolation rational function

$$\Lambda(x) := \sum_{i=0}^{N-1} \ell_i^{(t)}(x) R(x_i),$$

where

$$\ell_i^{(t)}(x) := \frac{\varphi_N^{(t)}(x) K_N^{(t)}(x_i) L_N(x_i)}{(x - x_i) K_N^{(t)}(x) L_N(x) \varphi_N'^{(t)}(x_i)}, \quad i = 0, 1, \dots, N-1.$$

It is readily shown that

$$\ell_i^{(t)}(x_j) = \delta_{i,j}, \quad i, j = 0, 1, \dots, N-1,$$

holds. Let

$$Q(x) := R(x) - \Lambda(x).$$

Then, the numerator of $Q(x)$ is a polynomial that has zeros at x_0, x_1, \dots, x_{N-1} . Since $R(x) \in \mathcal{D}(\mathcal{L}^{(t)})$, there exists a polynomial $P(x)$ such that

$$Q(x) = \frac{P(x) \varphi_N^{(t)}(x)}{K_N^{(t)}(x) L_N(x)}.$$

By the terminating condition (3.13b), we obtain

$$\begin{aligned} \mathcal{L}^{(t)}[R(x)] &= \mathcal{L}^{(t)}[\Lambda(x)] + \mathcal{L}^{(t)}[Q(x)] \\ &= \sum_{i=0}^{N-1} \mathcal{L}^{(t)}[\ell_i^{(t)}(x)] R(x_i) + \mathcal{L}^{(t)} \left[\frac{P(x) \varphi_N^{(t)}(x)}{K_N^{(t)}(x) L_N(x)} \right] \\ &= \sum_{i=0}^{N-1} \mathcal{L}^{(t)}[\ell_i^{(t)}(x)] R(x_i). \end{aligned}$$

Set $c_i^{(t)} := \mathcal{L}^{(t)}[\ell_i^{(t)}(x)]$, $i = 0, 1, \dots, N-1$, then the proof is completed. \square

Zhedanov [13] derived a formula to calculate the constants $c_0^{(t)}, c_1^{(t)}, \dots, c_{N-1}^{(t)}$. He used the second kind polynomials to derive it. Here, we give a direct calculation to check his result. From the quadrature formula (3.14), the moment is written as

$$\mu_m^{k,l,t} = \mathcal{L}^{(t)} \left[\frac{x^m}{K_k^{(t)}(x) L_l(x)} \right] = \sum_{i=0}^{N-1} \frac{c_i^{(t)} x_i^m}{K_k^{(t)}(x_i) L_l(x_i)}.$$

In the same manner as in Subsection 2.2, we thus obtain the following formulae for $j = 0, 1, \dots, N-1$:

$$\begin{aligned} \varphi_{N-1}^{(t)}(x_j) &= \frac{1}{\tau_{N-1}^{N-1, N-1, t}} \prod_{\substack{i=0 \\ i \neq j}}^{N-1} \frac{c_i^{(t)} (x_j - x_i)}{K_{N-1}^{(t)}(x_i) L_{N-1}(x_i)} \prod_{\substack{0 \leq \nu_0 < \nu_1 \leq N-1 \\ \nu_0 \neq j, \nu_1 \neq j}} (x_{\nu_1} - x_{\nu_0})^2, \\ \varphi_N'^{(t)}(x_j) &= \prod_{\substack{i=0 \\ i \neq j}}^{N-1} (x_j - x_i), \end{aligned}$$

and

$$\tau_N^{N-1,N-1,t} = \prod_{i=0}^{N-1} \frac{c_i^{(t)}}{K_{N-1}^{(t)}(x_i)L_{N-1}(x_i)} \prod_{\substack{0 \leq \nu_0 < \nu_1 \leq N-1 \\ \nu_0 \neq j, \nu_1 \neq j}} (x_{\nu_1} - x_{\nu_0})^2,$$

$$h_{N-1}^{(t)} = \frac{\tau_N^{N-1,N-1,t}}{\tau_{N-1}^{N-1,N-1,t}}.$$

Hence, we find the formula

$$c_i^{(t)} = \frac{h_{N-1}^{(t)} K_{N-1}^{(t)}(x_i) L_{N-1}(x_i)}{\varphi_{N-1}^{(t)}(x_i) \varphi_N^{(t)}(x_i)}, \quad i = 0, 1, \dots, N-1.$$

For the finite dimensional case, in the same manner as for the monic finite orthogonal polynomials (see Subsection 2.2), the characteristic polynomial is invariant under the time evolution:

$$\varphi_N^{(t+1)}(x) = \varphi_N^{(t)}(x).$$

From the results in Subsection 3.1, we can thus see that the solution to the initial value problem for the monic type finite R_{II} chain is given by

$$q_n^{(t)} = (s^{(t)} - \kappa_{t+n})^{-1} \frac{\tau_n^{n,n,t} \tau_{n+1}^{n,n+1,t+1}}{\tau_n^{n-1,n,t+1} \tau_{n+1}^{n+1,n+1,t}}, \quad e_n^{(t)} = (s^{(t)} - \kappa_{t+n}) \frac{\tau_{n-1}^{n-1,n-1,t+1} \tau_{n+1}^{n+1,n,t}}{\tau_n^{n,n-1,t} \tau_n^{n,n,t+1}},$$

where, because the moment is concretely given by

$$\mu_m^{k,l,t} = \sum_{i=0}^{N-1} \frac{c_i^{(0)} x_i^m \prod_{j=0}^{t-1} (x_i - s^{(j)})}{K_{t+k}(x_i) L_l(x_i)},$$

the expanded form of the Hankel determinant is

$$\tau_n^{k,l,t} = \sum_{0 \leq r_0 < r_1 < \dots < r_{n-1} \leq N-1} \left(\prod_{i=0}^{n-1} \frac{c_{r_i}^{(0)} \prod_{j=0}^{t-1} (x_{r_i} - s^{(j)})}{K_{t+k}(x_{r_i}) L_l(x_{r_i})} \prod_{0 \leq \nu_0 < \nu_1 \leq n-1} (x_{r_{\nu_1}} - x_{r_{\nu_0}})^2 \right).$$

The solution derived above yields the following theorem.

Theorem 3.3 (Convergence theorem for the monic type finite R_{II} chain). *Suppose that all the generalized eigenvalues x_0, x_1, \dots, x_{N-1} of the initial tridiagonal matrix pencil $(A^{(0)}, B^{(0)})$ are real, simple and arranged in descending order as $x_0 > x_1 > \dots > x_{N-1}$. Choose the parameters $s^{(t)}$ and κ_{t+N-1} as $x_{N-1} > s^{(t)}$ and $x_{N-1} \gg \kappa_{t+N-1}$ for all $t \geq 0$, respectively. Then, we have the asymptotics of the variables for $t \gg 0$:*

$$q_n^{(t)} = \frac{x_n - s^{(t)}}{s^{(t)} - \kappa_{t+n}} + O\left(\max\left\{\frac{\prod_{j=0}^t (x_n - s^{(j)})}{\prod_{j=0}^{t-1} (x_{n-1} - s^{(j)})} \frac{\prod_{j=0}^{t+n-1} (x_{n-1} - \kappa_j)}{\prod_{j=0}^{t+n-1} (x_n - \kappa_j)}, \frac{\prod_{j=0}^t (x_{n+1} - s^{(j)})}{\prod_{j=0}^{t-1} (x_n - s^{(j)})} \frac{\prod_{j=0}^{t+n-1} (x_n - \kappa_j)}{\prod_{j=0}^{t+n-1} (x_{n+1} - \kappa_j)}\right\}\right),$$

$$e_n^{(t)} = O\left(\frac{\prod_{j=0}^{t-1} (x_n - s^{(j)})}{\prod_{j=0}^t (x_{n-1} - s^{(j)})} \frac{\prod_{j=0}^{t+n-1} (x_{n-1} - \kappa_j)}{\prod_{j=0}^{t+n} (x_n - \kappa_j)}\right).$$

Hence, the variables $q_n^{(t)}$ and $e_n^{(t)}$ converge to $\frac{x_n - s^{(t)}}{s^{(t)} - \kappa_{t+n}}$ and 0 as $t \rightarrow +\infty$, respectively.

This theorem implies that, from (3.6), the elements $v_n^{(t)}$ and $w_n^{(t)}$ of the tridiagonal matrices $A^{(t)}$ and $B^{(t)}$ converge to x_n and 0 as $t \rightarrow +\infty$, respectively. Further, we can see that the parameters $s^{(t)}$ and κ_{t+n} determine the convergence speed; the parameter $s^{(t)}$ works as the origin shift, which is the same as for the dqds algorithm (see the end of Subsection 2.3).

Next, we discuss the matrix form of the monic type finite R_{II} chain. Introduce the rational functions defined by the following three-term recurrence relation:

$$\begin{aligned} \Phi_{-1}^{(t)}(x) &:= 0, \quad \Phi_0^{(t)}(x) := 1, \\ (x - \kappa_{t+n})\Phi_{n+1}^{(t)}(x) &:= -\left((1 + w_n^{(t)})x - v_n^{(t)}\right)\Phi_n^{(t)}(x) - w_n^{(t)}(x - \lambda_n)\Phi_{n-1}^{(t)}(x), \quad n = 0, 1, \dots, N-1. \end{aligned} \quad (3.15)$$

By comparing to the three-term recurrence relation (3.1), the relation

$$\Phi_n^{(t)}(x) = \frac{\varphi_n^{(t)}(x)}{K_n^{(t)}(x)}, \quad n = 0, 1, \dots, N,$$

is verified. Let

$$\Phi^{(t)}(x) := \begin{pmatrix} \Phi_0^{(t)}(x) \\ \Phi_1^{(t)}(x) \\ \vdots \\ \Phi_{N-1}^{(t)}(x) \end{pmatrix}, \quad \Phi_N^{(t)}(x) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Phi_N^{(t)}(x) \end{pmatrix}.$$

Then, the three-term recurrence relation (3.15) is rewritten as

$$A^{(t)}\Phi^{(t)}(x) + \kappa_{t+N}\Phi_N^{(t)}(x) = x\left(B^{(t)}\Phi^{(t)}(x) + \Phi_N^{(t)}(x)\right). \quad (3.16a)$$

Further, let $L_A^{(t)}$, $L_B^{(t)}$, and $R^{(t)}$ be bidiagonal matrices:

$$\begin{aligned} L_A^{(t)} &:= \begin{pmatrix} \kappa_t & & & & \\ -\lambda_1 e_1^{(t)} & \kappa_{t+1} & & & \\ & -\lambda_2 e_2^{(t)} & \ddots & & \\ & & \ddots & \ddots & \\ & & & -\lambda_{N-1} e_{N-1}^{(t)} & \kappa_{t+N-1} \end{pmatrix}, \quad L_B^{(t)} := \begin{pmatrix} 1 & & & & \\ -e_1^{(t)} & 1 & & & \\ & -e_2^{(t)} & \ddots & & \\ & & \ddots & \ddots & \\ & & & -e_{N-1}^{(t)} & 1 \end{pmatrix}, \\ R^{(t)} &:= \begin{pmatrix} q_0^{(t)} & -1 & & & \\ & q_1^{(t)} & -1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ & & & & q_{N-1}^{(t)} \end{pmatrix}, \end{aligned}$$

and $D_q^{(t)}$, $D_e^{(t)}$ and $\hat{D}_e^{(t)}$ be diagonal matrices:

$$\begin{aligned} D_q^{(t)} &:= \text{diag}\left(1 + q_0^{(t)}, 1 + q_1^{(t)}, \dots, 1 + q_{N-1}^{(t)}\right), \\ D_e^{(t)} &:= \text{diag}\left(1, 1 + e_1^{(t)}, \dots, 1 + e_{N-1}^{(t)}\right), \quad \hat{D}_e^{(t)} := \text{diag}\left(1 + e_1^{(t)}, \dots, 1 + e_{N-1}^{(t)}, 1\right). \end{aligned}$$

Then, the spectral transformations (3.4) are written in terms of the rational functions $\{\Phi_n^{(t)}(x)\}_{n=0}^N$ as

$$(x - s^{(t)})D_q^{(t)}\Phi^{(t+1)}(x) = (x - \kappa_t)\left(R^{(t)}\Phi^{(t)}(x) - \Phi_N^{(t)}(x)\right), \quad (3.16b)$$

$$D_e^{(t)}\Phi^{(t)}(x) = (x - \kappa_t)^{-1}\left(xL_B^{(t)} - L_A^{(t)}\right)\Phi^{(t+1)}(x). \quad (3.16c)$$

Equations (3.16) yield

$$\begin{aligned}
& x \left(B^{(t+1)} \Phi^{(t+1)}(x) + \Phi_N^{(t+1)}(x) \right) - A^{(t+1)} \Phi^{(t+1)}(x) - \kappa_{t+N} \Phi_N^{(t+1)}(x) \\
&= x \left(\left(-D_q^{(t+1)} L_B^{(t+1)} (D_q^{(t+1)})^{-1} R^{(t+1)} + D_q^{(t+1)} D_e^{(t+1)} \right) \Phi^{(t+1)}(x) + \Phi_N^{(t+1)}(x) \right) \\
&\quad - \left(-D_q^{(t+1)} L_A^{(t+1)} (D_q^{(t+1)})^{-1} R^{(t+1)} + s^{(t+1)} D_q^{(t+1)} D_e^{(t+1)} \right) \Phi^{(t+1)}(x) - \kappa_{t+N} \Phi_N^{(t+1)}(x) \\
&= x \left(\left(-\hat{D}_e^{(t)} R^{(t)} (D_e^{(t)})^{-1} L_B^{(t)} + D_q^{(t)} \hat{D}_e^{(t)} \right) \Phi^{(t+1)}(x) + \Phi_N^{(t+1)}(x) \right) \\
&\quad - \left(-\hat{D}_e^{(t)} R^{(t)} (D_e^{(t)})^{-1} L_A^{(t)} + s^{(t)} D_q^{(t)} \hat{D}_e^{(t)} \right) \Phi^{(t+1)}(x) - \kappa_{t+N} \Phi_N^{(t+1)}(x) \\
&= \mathbf{0}.
\end{aligned}$$

Hence, the compatibility condition for (3.16), i.e. the matrix form of the monic type finite R_{II} chain, is given by

$$\begin{aligned}
A^{(t+1)} &= -D_q^{(t+1)} L_A^{(t+1)} (D_q^{(t+1)})^{-1} R^{(t+1)} + s^{(t+1)} D_q^{(t+1)} D_e^{(t+1)} \\
&= -\hat{D}_e^{(t)} R^{(t)} (D_e^{(t)})^{-1} L_A^{(t)} + s^{(t)} D_q^{(t)} \hat{D}_e^{(t)}, \\
B^{(t+1)} &= -D_q^{(t+1)} L_B^{(t+1)} (D_q^{(t+1)})^{-1} R^{(t+1)} + D_q^{(t+1)} D_e^{(t+1)} \\
&= -\hat{D}_e^{(t)} R^{(t)} (D_e^{(t)})^{-1} L_B^{(t)} + D_q^{(t)} \hat{D}_e^{(t)}.
\end{aligned}$$

This leads to

$$\begin{aligned}
A^{(t+1)} &= \hat{D}_e^{(t)} R^{(t)} (D_q^{(t)} D_e^{(t)})^{-1} A^{(t)} (R^{(t)})^{-1} D_q^{(t)}, \\
B^{(t+1)} &= \hat{D}_e^{(t)} R^{(t)} (D_q^{(t)} D_e^{(t)})^{-1} B^{(t)} (R^{(t)})^{-1} D_q^{(t)},
\end{aligned}$$

and

$$x B^{(t+1)} - A^{(t+1)} = \hat{D}_e^{(t)} R^{(t)} (D_q^{(t)} D_e^{(t)})^{-1} (x B^{(t)} - A^{(t)}) (R^{(t)})^{-1} D_q^{(t)}.$$

The last equation implies that the generalized eigenvalues of the tridiagonal matrix pencil $(A^{(t)}, B^{(t)})$ are conserved under the time evolution.

4. Generalized eigenvalue algorithm

4.1. Subtraction-free form of the monic type R_{II} chain

In Section 3, we have presented the convergence theorem for the monic type finite R_{II} chain (Theorem 3.3). This theorem allows us to design a generalized eigenvalue algorithm for tridiagonal matrix pencils.

Consider a pair of tridiagonal matrices of order N as input:

$$A = \begin{pmatrix} a_{0,0} & a_{0,1} & & & & \\ a_{1,0} & a_{1,1} & a_{1,2} & & & \\ & a_{2,1} & \ddots & \ddots & & \\ & & \ddots & \ddots & a_{N-2,N-1} & \\ & & & a_{N-1,N-2} & a_{N-1,N-1} & \end{pmatrix}, \quad B = \begin{pmatrix} b_{0,0} & b_{0,1} & & & & \\ b_{1,0} & b_{1,1} & b_{1,2} & & & \\ & b_{2,1} & \ddots & \ddots & & \\ & & \ddots & \ddots & b_{N-2,N-1} & \\ & & & b_{N-1,N-2} & b_{N-1,N-1} & \end{pmatrix}. \quad (4.1)$$

Suppose that all the subdiagonal elements $b_{0,1}, b_{1,2}, \dots, b_{N-2,N-1}$ and $b_{1,0}, b_{2,0}, \dots, b_{N-1,N-2}$ of the matrix B are nonzero, and all the leading principal minors of the matrix B are nonzero. Then, the transformation

$$A^{(0)} := V_1 U A U^{-1} V_2, \quad B^{(0)} := V_1 U B U^{-1} V_2$$

gives the initial matrix pencil of the form (3.11) for the monic type finite R_{II} chain, where

$$U := \text{diag}(1, b_{0,1}, b_{0,1}b_{1,2}, \dots, b_{0,1}b_{1,2} \dots b_{N-2,N-1}),$$

$$V_1 := \text{diag}((\det B_1)^{-1}, (\det B_2)^{-1}, \dots, (\det B_N)^{-1}), \quad V_2 := \text{diag}(1, \det B_1, \det B_2, \dots, \det B_{N-1}),$$

and B_n is the n -th order leading principal submatrix of the matrix B . Namely, the elements of $A^{(0)}$ and $B^{(0)}$ are computed by

$$v_n^{(0)} := a_{n,n} \frac{\det B_n}{\det B_{n+1}}, \quad w_n^{(0)} := b_{n-1,n} b_{n,n-1} \frac{\det B_{n-1}}{\det B_{n+1}}, \quad \kappa_n := \frac{a_{n,n+1}}{b_{n,n+1}}, \quad \lambda_n := \frac{a_{n,n-1}}{b_{n,n-1}}. \quad (4.2)$$

Note that, if n is large, an overflow may occur when one computes $\det B_n$ directly. The values $\frac{\det B_n}{\det B_{n+1}}$ and $\frac{\det B_{n-1}}{\det B_{n+1}}$ should be computed by the LU decomposition. Next, by the relation (3.6), “decompose” the matrix pencil $(A^{(0)}, B^{(0)})$ to the variables of the monic type finite R_{II} chain:

$$e_0^{(0)} := 0, \quad e_N^{(0)} := 0, \quad (4.3a)$$

$$\tilde{e}_n^{(0)} := \frac{w_n^{(0)}}{q_{n-1}^{(0)}}, \quad e_n^{(0)} := \tilde{e}_n^{(0)} \frac{1 + q_{n-1}^{(0)}}{1 + q_n^{(0)}}, \quad n = 1, 2, \dots, N-1, \quad (4.3b)$$

$$q_n^{(0)} := \frac{v_n^{(0)} - s^{(0)}(1 + w_n^{(0)}) - (s^{(0)} - \lambda_n)\tilde{e}_n^{(0)}}{s^{(0)} - \kappa_n}, \quad n = 0, 1, \dots, N-1. \quad (4.3c)$$

Notice that the initial matrix pencil $(A^{(0)}, B^{(0)})$ does not fix the values of the parameters $s^{(0)}$ and κ_{N-1} . We must choose the parameters $s^{(0)}$ and κ_{N-1} appropriately. We will discuss how to choose the parameters in the end of this subsection. After that, compute the time evolution of the monic type finite R_{II} chain by using (3.12) iteratively; i.e., for each $t \geq 0$, compute

$$e_0^{(t+1)} := 0, \quad e_N^{(t+1)} := 0, \quad (4.4a)$$

$$e_n^{(t+1)} := e_n^{(t)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} \frac{1 + q_{n-1}^{(t+1)}}{1 + q_n^{(t+1)}} \frac{1 + e_{n+1}^{(t)}}{1 + e_n^{(t)}}, \quad n = 1, 2, \dots, N-1, \quad (4.4b)$$

$$q_n^{(t+1)} := (s^{(t+1)} - \kappa_{t+n+1})^{-1} \left((s^{(t+1)} - \kappa_{t+n}) q_n^{(t)} \frac{1 + e_{n+1}^{(t)}}{1 + e_n^{(t)}} - (s^{(t+1)} - \lambda_n) e_n^{(t+1)} \frac{1 + q_n^{(t+1)}}{1 + q_{n-1}^{(t+1)}} \right. \\ \left. + (s^{(t+1)} - \lambda_{n+1}) e_{n+1}^{(t)} - (s^{(t+1)} - s^{(t)})(1 + q_n^{(t)})(1 + e_{n+1}^{(t)}) \right), \quad n = 0, 1, \dots, N-1. \quad (4.4c)$$

Here, we also have to choose the parameters $s^{(t+1)}$ and κ_{t+N} for computing the above recurrence equations. From the results in Subsection 3.2, we can see that if the absolute values of all the subdiagonal elements $\lambda_n w_n^{(t)}$ and $w_n^{(t)}$ of the matrix pencil $(A^{(t)}, B^{(t)})$ become sufficiently small at a time t , then the values $(s^{(t)} - \kappa_{t+n}) q_n^{(t)} + s^{(t)}$ give the generalized eigenvalues of the initial tridiagonal matrix pencil (A, B) . In general, however, equation (4.4c) requires subtraction operations, which may degrade the accuracy by the loss of significant digits. A subtraction-free form of the monic type finite R_{II} chain may resolve the problem.

Let us introduce an auxiliary variable

$$d_n^{(t+1)} = \frac{(s^{(t+1)} - \kappa_{t+n+1}) q_n^{(t+1)} - (s^{(t+1)} - \lambda_{n+1}) e_{n+1}^{(t)}}{1 + e_{n+1}^{(t)}}, \quad n = 0, 1, \dots, N-1.$$

This is an analogue of the auxiliary variable (2.22) introduced in the dqds algorithm. Then, the subtraction-free form is derived as

$$d_0^{(t+1)} := (s^{(t)} - \kappa_t)q_0^{(t)} - (s^{(t+1)} - s^{(t)}), \quad d_n^{(t+1)} := d_{n-1}^{(t+1)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} - (s^{(t+1)} - s^{(t)})(1 + q_n^{(t)}),$$

$$n = 1, 2, \dots, N-1, \quad (4.5a)$$

$$q_n^{(t+1)} := \frac{(s^{(t+1)} - \lambda_{n+1})e_{n+1}^{(t)} + d_n^{(t+1)}(1 + e_{n+1}^{(t)})}{s^{(t+1)} - \kappa_{t+n+1}}, \quad n = 0, 1, \dots, N-1, \quad (4.5b)$$

$$e_0^{(t+1)} := 0, \quad e_n^{(t+1)} := e_n^{(t)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} \frac{1 + q_{n-1}^{(t+1)}}{1 + q_n^{(t)}} \frac{1 + e_{n+1}^{(t)}}{1 + e_n^{(t)}}, \quad n = 1, 2, \dots, N-1, \quad e_N^{(t+1)} := 0. \quad (4.5c)$$

From the spectral transformations (3.4), we have

$$-(1 + e_{n+1}^{(t-1)})\varphi_{n+1}^{(t-1)}(s^{(t)}) = \left((s^{(t)} - \kappa_{t+n})q_n^{(t)} - (s^{(t)} - \lambda_{n+1})e_{n+1}^{(t-1)} \right) \varphi_n^{(t)}(s^{(t)})$$

$$\Rightarrow d_n^{(t)} = -\frac{\varphi_{n+1}^{(t-1)}(s^{(t)})}{\varphi_n^{(t)}(s^{(t)})} = \frac{\tau_n^{n,n,t} \sigma_{n+1}^{n,n+1,t}}{\tau_{n+1}^{n+1,n+1,t-1} \tau_n^{n-1,n,t+1}},$$

where

$$\sigma_n^{k,l,t} := |\mu_{i+j+1}^{k+1,l,t-1} - s^{(t)} \mu_{i+j}^{k+1,l,t-1}|_{0 \leq i,j \leq n-1}.$$

In addition, we already have the expression of $q_n^{(t)}$ (3.5). Hence, we obtain a sufficient condition for computing the recurrence equation (4.5) without subtraction operations except the shift terms in (4.5a): for all n and t ,

$$w_n^{(0)} > 0, \quad (4.6a)$$

$$(-1)^n \varphi_n^{(t)}(s^{(t)}) = \det(A_n^{(t)} - s^{(t)} B_n^{(t)}) > 0, \quad (4.6b)$$

$$(-1)^n \varphi_n^{(t)}(s^{(t+1)}) = \det(A_n^{(t)} - s^{(t+1)} B_n^{(t)}) > 0, \quad (4.6c)$$

$$s^{(t)} > \kappa_{t+n}, \quad s^{(t)} > \lambda_n. \quad (4.6d)$$

By (4.2), if the input tridiagonal matrix B is a real symmetric positive (or negative) definite matrix, then the condition (4.6a) is satisfied. Further, assume that the generalized eigenvalues x_0, x_1, \dots, x_{N-1} of the input tridiagonal matrix pencil (A, B) are all real and simple, the matrix A is a real matrix and the conditions $w_n^{(t)} > 0$ and $\kappa_{t+n-1} = \lambda_n$ are satisfied for $n = 1, 2, \dots, N-1$ at some time t . Then, it is shown that if the parameter $s^{(t)}$ is chosen as $s^{(t)} < \min\{x_0, x_1, \dots, x_{N-1}\}$, the condition (4.6b) is satisfied. The condition (4.6c) is also satisfied with $s^{(t+1)} < \min\{x_0, x_1, \dots, x_{N-1}\}$. From Theorem 3.3, if $s^{(t)}$ is chosen as close as possible to $\min\{x_0, x_1, \dots, x_{N-1}\}$ under the conditions (4.6), the convergence speed is accelerated.

By summarizing this subsection, Algorithm 1 is proposed as a new generalized eigenvalue algorithm for tridiagonal matrix pencils based on the monic type finite R_{II} chain.

4.2. Numerical examples

We shall give numerical examples. To construct test problems with known generalized eigenvalues, let us consider the monic finite orthogonal polynomials $\{p_n(x)\}_{n=0}^N$ defined by

$$p_{n+1}(x) := \left(x - \frac{N-1}{2}\right) p_n(x) - \frac{n(N-n)}{4} p_{n-1}(x), \quad n = 0, 1, \dots, N-1,$$

Algorithm 1 The proposed generalized eigenvalue algorithm based on the monic type finite R_{II} chain

```
1: function GEVRII( $A, B$ ) ▷  $A$  and  $B$  are tridiagonal matrices of the form (4.1)
2:   Compute  $\{v_n^{(0)}\}_{n=0}^{N-1}$ ,  $\{w_n^{(0)}\}_{n=1}^{N-1}$ ,  $\{\kappa_n\}_{n=0}^{N-2}$ , and  $\{\lambda_n\}_{n=1}^{N-1}$  by (4.2)
3:   Set the parameters  $s^{(0)}$  and  $\kappa_{N-1}$  appropriately ▷ See Theorem 3.3 and the condition (4.6)
4:   Compute  $\{q_n^{(0)}\}_{n=0}^{N-1}$  and  $\{e_n^{(0)}\}_{n=0}^N$  by (4.3)
5:    $t := 0$ 
6:   repeat
7:     Set the parameters  $s^{(t+1)}$  and  $\kappa_{t+N}$  appropriately ▷ See Theorem 3.3 and the condition (4.6)
8:     Compute  $\{q_n^{(t+1)}\}_{n=0}^{N-1}$  and  $\{e_n^{(t+1)}\}_{n=0}^N$  by (4.5)
9:      $t := t + 1$ 
10:    for  $n = 1, 2, \dots, N - 1$  do
11:       $w_n^{(t)} := q_{n-1}^{(t)} e_n^{(t)} \frac{1+q_n^{(t)}}{1+q_{n-1}^{(t)}}$ 
12:    end for
13:    until the absolute values of  $w_n^{(t)}$  and  $\lambda_n w_n^{(t)}$  are sufficiently small for all  $n = 1, 2, \dots, N - 1$ 
14:    return  $\{(s^{(t)} - \kappa_{t+n})q_n^{(t)} + s^{(t)}\}_{n=0}^{N-1}$ 
15: end function
```

with $p_{-1}(x) := 0$ and $p_0(x) := 1$. The polynomials $\{p_n(x)\}_{n=0}^N$ are the monic Krawtchouk polynomials with a special parameter and it is well known that the Krawtchouk polynomials are orthogonal on $x = 0, 1, \dots, N - 1$ with respect to the binomial distribution [27]. This means that the tridiagonal matrix of order N

$$\tilde{K}_N := \begin{pmatrix} (N-1)/2 & 1 & & & & \\ (N-1)/4 & (N-1)/2 & 1 & & & \\ & 2(N-2)/4 & (N-1)/2 & 1 & & \\ & & 3(N-3)/4 & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & (N-1)/4 & (N-1)/2 \end{pmatrix} \in \mathbb{R}^{N \times N}$$

has the eigenvalues $0, 1, \dots, N - 1$. The symmetric tridiagonal matrix

$$K_N := \begin{pmatrix} (N-1)/2 & \sqrt{(N-1)/4} & & & & \\ \sqrt{(N-1)/4} & (N-1)/2 & \sqrt{2(N-2)/4} & & & \\ & \sqrt{2(N-2)/4} & (N-1)/2 & \sqrt{3(N-3)/4} & & \\ & & \sqrt{3(N-3)/4} & \ddots & \ddots & \\ & & & \ddots & \ddots & \sqrt{(N-1)/4} \\ & & & & \sqrt{(N-1)/4} & (N-1)/2 \end{pmatrix} \in \mathbb{R}^{N \times N}$$

is similar to \tilde{K}_N . Hence, it is readily shown that the tridiagonal matrix pencil $(K_N + 2I_N, K_N + I_N)$ has the generalized eigenvalues $(n+1)/n$, $n = 1, 2, \dots, N$.

The following experiments were run on a Linux PC with kernel 3.7.4 and gcc 4.7.2 on Intel Core i5 760 2.80 GHz CPU and 4 GB memory. All the computations were performed in double precision and the stopping criterion (line 13 in Algorithm 1) was $|w_n^{(t)}| < 10^{-20}$ and $|\lambda_n w_n^{(t)}| < 10^{-20}$ for all $n = 1, 2, \dots, N - 1$.

Example 1. The first example is the case of $N = 5$:

$$K_5 = \begin{pmatrix} 2 & 1 & & & \\ 1 & 2 & \sqrt{3/2} & & \\ & \sqrt{3/2} & 2 & \sqrt{3/2} & \\ & & \sqrt{3/2} & 2 & 1 \\ & & & 1 & 2 \end{pmatrix}.$$

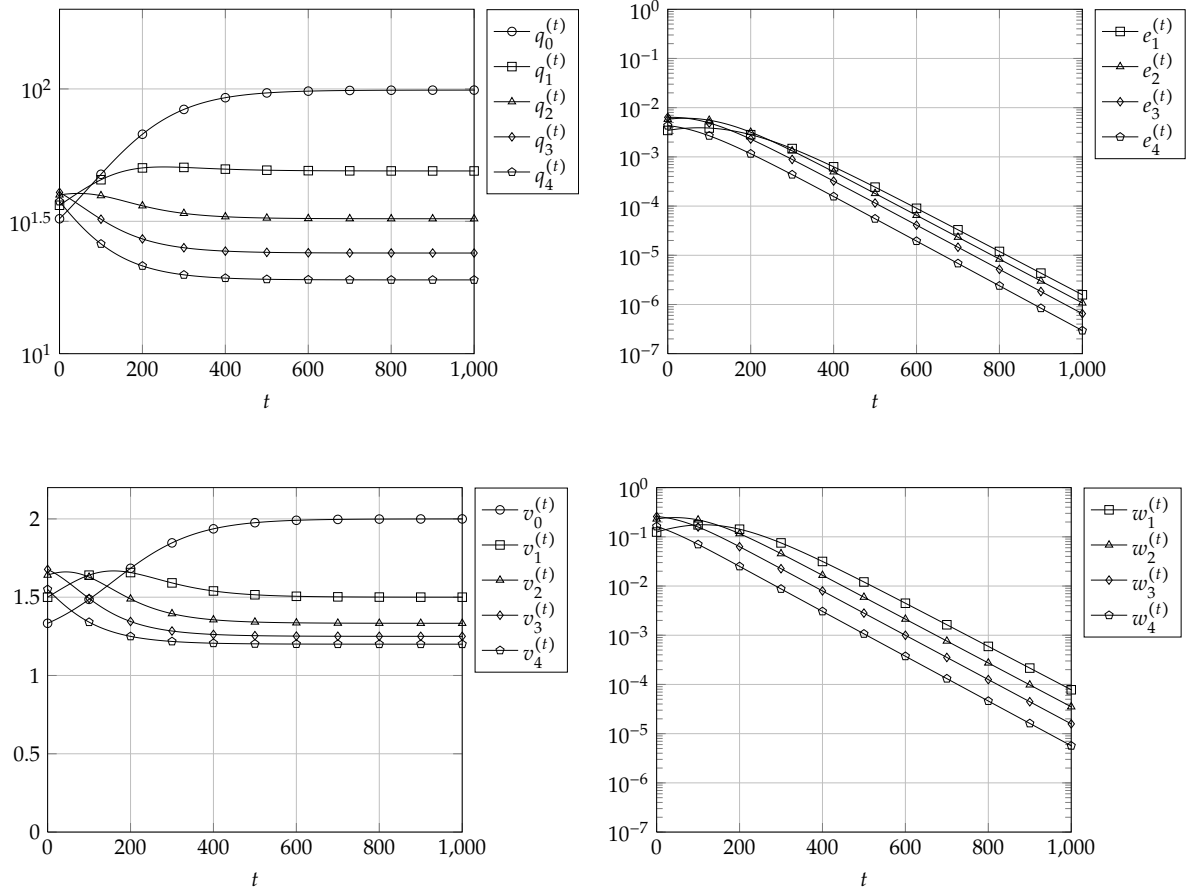


Figure 1: The behaviour of the variables of the monic type finite R_{II} chain for the input tridiagonal matrix pencil $(K_5 + 2I_5, K_5 + I_5)$ with the parameters $s^{(t)} = 1.01$ for all $t \geq 0$ and $\kappa_n = 1$ for all $n \geq 4$.

The generalized eigenvalues of the matrix pencil $(K_5 + 2I_5, K_5 + I_5)$ are 2, $3/2$, $4/3$, $5/4$, and $6/5$. By this example, we will observe the behaviour of the variables of the monic type finite R_{II} chain and confirm that the proposed algorithm computes the generalized eigenvalues of a given matrix pencil and its convergence speed depends on the parameters $s^{(t)}$ and κ_{t+n} .

Figure 1 shows the result with the parameters $s^{(t)} = 1.01$ for all $t \geq 0$ and $\kappa_n = 1$ for all $n \geq 4$, where $q_n^{(t)}$ and $e_n^{(t)}$ are the variables of the monic type R_{II} chain, $v_n^{(t)}$ are the diagonal elements of $A^{(t)}$, and $w_n^{(t)}$ are the subdiagonal elements of $B^{(t)}$ (see equations (3.6b) and (3.6c)). We can confirm that $v_n^{(t)}$ and $w_n^{(t)}$ converge linearly to the eigenvalues and zero, respectively. Since the shift parameter $s^{(t)}$ is not so close to the minimal eigenvalue $6/5 = 1.2$, the stopping criterion is satisfied at $t = 4605$.

Figure 2 shows the result with more suitable parameters: $s^{(t)} = 1.19$ for all $t \geq 0$ and $\kappa_n = -10000$ for all $n \geq 4$. The convergence speed is much faster than the former example; the stopping criterion is satisfied at $t = 48$. Table 1 shows the computed eigenvalues.

Example 2. Next, the test cases for $N = 512, 1024, 2048, 4096, 8192$ were computed by two methods. By these examples, we will compare the computation time and the accuracy of the proposed algorithm with a routine called DSYGV in LAPACK 3.4.2 [28]. DSYGV computes the generalized eigenvalues of a given matrix pencil (A, B) in double precision, where A is real symmetric and B is real symmetric and positive definite. Internally, DSYGV computes the Cholesky factorization $B = LL^T$, where L is a lower triangular matrix, trans-

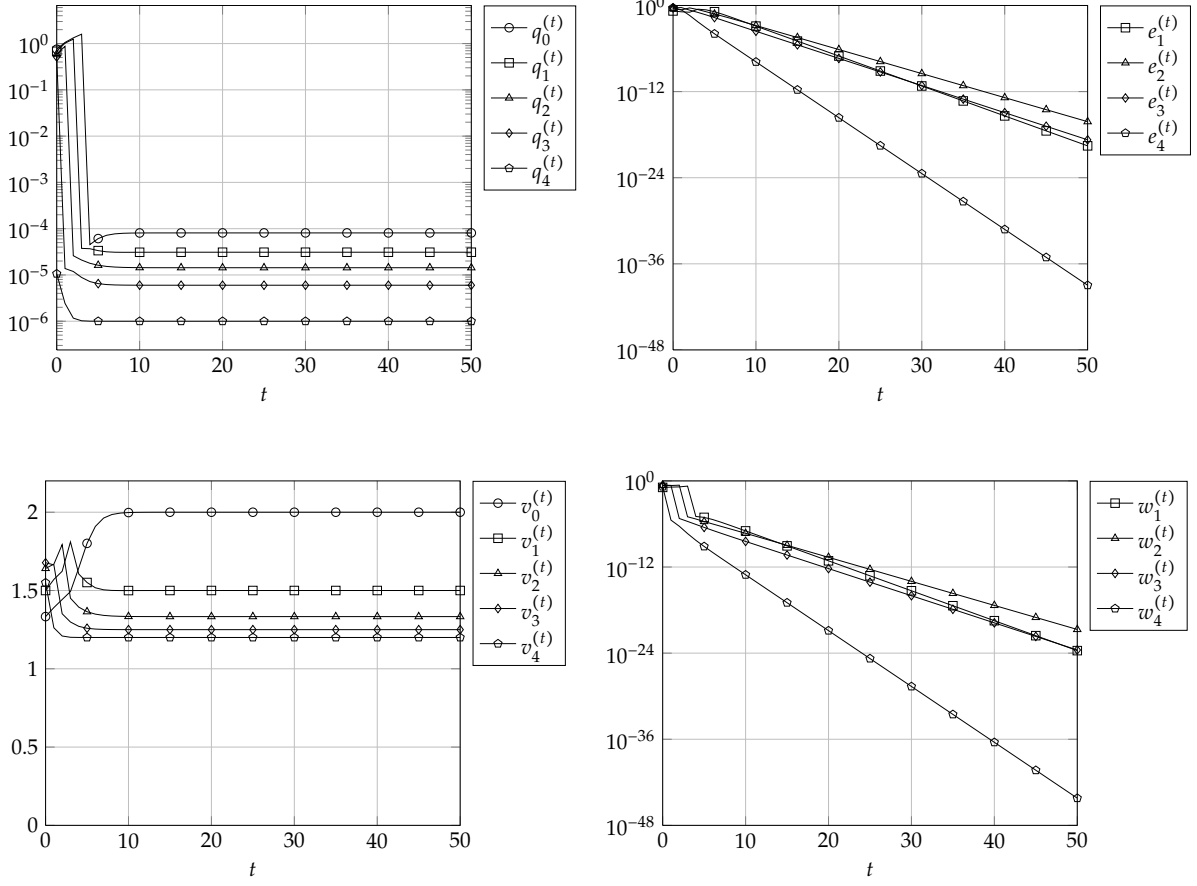


Figure 2: The behaviour of the variables of the monic type finite R_{II} chain for the input tridiagonal matrix pencil $(K_5 + 2I_5, K_5 + I_5)$ with the parameters $s^{(t)} = 1.19$ for all $t \geq 0$ and $\kappa_n = -10000$ for all $n \geq 4$.

Table 1: The eigenvalues computed by Algorithm 1. The parameters are $s^{(t)} = 1.19$ and $\kappa_n = -10000$ for all $t \geq 0$ and $n \geq 4$.

Computed eigenvalues	True eigenvalues
1.9999999999999998	2.0000000000000000
1.4999999999999991	1.5000000000000000
1.3333333333333335	1.3333333333333333
1.2500000000000000	1.2500000000000000
1.2000000000000000	1.2000000000000000

Table 2: The results of the computation by Algorithm 1 for the generalized eigenvalue problems of $(K_N + 2I_N, K_N + I_N)$.

Problem size (N)	512	1024	2048	4096	8192
Computation time [sec.]	0.0958	0.392	1.58	6.24	24.6
Maximum relative error	3.109×10^{-15}	3.405×10^{-15}	1.776×10^{-15}	3.701×10^{-15}	2.043×10^{-14}
Average relative error	1.344×10^{-16}	1.211×10^{-16}	1.154×10^{-16}	1.072×10^{-16}	1.129×10^{-16}

Table 3: The results of the computation by DSYGV in LAPACK for the generalized eigenvalue problems of $(K_N + 2I_N, K_N + I_N)$.

Problem size (N)	512	1024	2048	4096	8192
Computation time [sec.]	0.162	1.92	30.3	307	2400
Maximum relative error	3.664×10^{-15}	6.815×10^{-15}	1.304×10^{-14}	1.684×10^{-14}	2.949×10^{-14}
Average relative error	6.673×10^{-16}	8.469×10^{-16}	1.035×10^{-15}	1.276×10^{-15}	1.508×10^{-15}

forms the generalized eigenvalue problem $A\boldsymbol{\varphi} = xB\boldsymbol{\varphi}$ to the eigenvalue problem $L^{-1}AL^{-T}(L^T\boldsymbol{\varphi}) = x(L^T\boldsymbol{\varphi})$ and solves the eigenvalue problem. We should remark that, even if A and B are both tridiagonal, $L^{-1}AL^{-T}$ is a dense matrix in general. Hence, we expect that DSYGV spends much time for large problems. On the other hand, the proposed algorithm preserves the tridiagonal form of the matrices $A^{(t)}$ and $B^{(t)}$. The proposed algorithm will thus compute the generalized eigenvalues of tridiagonal matrix pencils fast and accurately for large problems.

Tables 2 and 3 show the results of the computation by the proposed algorithm and DSYGV, respectively. The parameters for the proposed algorithm are $s^{(t)} = (N+2)/(N+1)$ for all $t \geq 0$ and $\kappa_n = -10000$ for all $n \geq N-1$. In all the cases, the proposed algorithm is faster and more accurate than DSYGV. In particular, the proposed algorithm has an advantage in computation time for large problems. Remark that the techniques called deflation and splitting (if $|w_n^{(t)}|$ and $|\lambda_n w_n^{(t)}|$ become sufficiently small for some n at a time t , then the problem can be deflated or split into two problems) were not implemented in the program used for the experiments. These techniques may further accelerate the proposed algorithm.

5. Conclusion

In this paper, we have studied the monic type R_{II} chain in detail and proposed a generalized eigenvalue algorithm for tridiagonal matrix pencils based on a subtraction-free form of the monic type finite R_{II} chain. It has been shown that, similarly to the dqds algorithm, the parameter $s^{(t)}$ in the monic type finite R_{II} chain plays the role of the origin shifts to accelerate convergence and the proposed algorithm computes the generalized eigenvalues of tridiagonal matrix pencils fast and accurately.

In Example 2, the shift parameter $s^{(t)}$ is chosen ideally and all the conditions (4.6) are satisfied. However, it is difficult to make this situation in general. Further improvements are thus required for practical use. First, in general, the condition for positivity (4.6) is not sufficient for applications; the condition does not provide concrete ways to choose the parameters for general cases. Second, for applying the proposed algorithm to general (not tridiagonal) matrix pencils, a preconditioning called simultaneous tridiagonalization (see, e.g., [29, 30]) is required. In addition to the improvements, comparisons with traditional methods should be discussed. These are left for future work.

Acknowledgments

The authors would like to thank Professor Yoshimasa Nakamura and Professor Alexei Zhedanov for valuable discussions and comments. This work was supported by JSPS KAKENHI Grant Numbers 11J04105 and 22540224.

References

- [1] C. Brezinski, Convergence acceleration during 20th century, *J. Comput. Appl. Math.* 122 (2000) 1–21.
- [2] Y. Nakamura, A new approach to numerical algorithms in terms of integrable systems, in: *Proceedings of the 12th International Conference on Informatics Research for Development of Knowledge Society Infrastructure (ICKS'04)*, IEEE Society Press, 2004, pp. 194–205.
- [3] M. T. Chu, Linear algebra algorithms as dynamical systems, *Acta Numer.* 17 (2008) 1–86.
- [4] S. Tsujimoto, Y. Nakamura, M. Iwasaki, The discrete Lotka-Volterra system computes singular values, *Inverse Problems* 17 (2001) 53–58.
- [5] M. Iwasaki, Y. Nakamura, Accurate computation of singular values in terms of shifted integrable schemes, *Japan J. Indust. Appl. Math.* 23 (2006) 239–259.
- [6] Y. Minesaki, Y. Nakamura, The discrete relativistic Toda molecule equation and a Padé approximation algorithm, *Numer. Algorithms* 27 (2001) 219–235.
- [7] A. Mukaihiro, Y. Nakamura, Schur flow for orthogonal polynomials on the unit circle and its integrable discretization, *J. Comput. Appl. Math.* 139 (2002) 75–94.
- [8] A. Fukuda, E. Ishiwata, M. Iwasaki, Y. Nakamura, The discrete hungry Lotka-Volterra system and a new algorithm for computing matrix eigenvalues, *Inverse Problems* 25 (2009) 1–17.
- [9] A. Fukuda, Y. Yamamoto, M. Iwasaki, E. Ishiwata, Y. Nakamura, On a shifted LR transformation derived from the discrete hungry toda equation, *Monatsh. Math. Online* first article.
- [10] H. Sekido, An algorithm for calculating D -optimal designs for polynomial regression through a fixed point, *J. Stat. Plann. Inference* 142 (2012) 935–943.
- [11] H. Sekido, An algorithm for calculating D -optimal designs for trigonometric regression through given points in terms of the discrete modified KdV equation, *J. Math-for-Indust.* 4 (2012) 17–23.
- [12] V. Spiridonov, A. Zhedanov, Spectral transformation chains and some new biorthogonal rational functions, *Comm. Math. Phys.* 210 (2000) 49–83.
- [13] A. Zhedanov, Biorthogonal rational functions and the generalized eigenvalue problem, *J. Approx. Theory* 101 (1999) 303–329.
- [14] K. V. Fernando, B. N. Parlett, Accurate singular values and differential qd algorithms, *Numer. Math.* 67 (1994) 191–229.
- [15] V. Papageorgiou, B. Grammaticos, A. Ramani, Orthogonal polynomial approach to discrete Lax pairs for initial boundary-value problems of the QD algorithm, *Lett. Math. Phys.* 34 (1995) 91–101.
- [16] V. Spiridonov, A. Zhedanov, Discrete Darboux transformations, the discrete-time Toda lattice, and the Askey-Wilson polynomials, *Methods Appl. Anal.* 2 (1995) 369–398.
- [17] M. E. H. Ismail, D. R. Masson, Generalized orthogonality and continued fractions, *J. Approx. Theory* 83 (1995) 1–40.
- [18] T. Tokihiro, D. Takahashi, J. Matsukidaira, J. Satsuma, From soliton equations to integrable cellular automata through a limiting procedure, *Phys. Rev. Lett.* 76 (1996) 3247–3250.
- [19] I. Itenberg, G. Mikhalkin, E. Shustin, *Tropical Algebraic Geometry*, 2nd Edition, Birkhäuser Verlag, Basel–Boston–Berlin, 2009.
- [20] K. Maeda, S. Tsujimoto, Box-ball systems related to the nonautonomous ultradiscrete Toda equation on the finite lattice, *JSIAM Lett.* 2 (2010) 95–98.
- [21] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach Science Publishers, New York–London–Paris, 1978.
- [22] A. Zhedanov, Rational spectral transformations and orthogonal polynomials, *J. Comput. Appl. Math.* 85 (1997) 67–86.
- [23] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, Cambridge, 2004, translated from the original Japanese book by A. Nagai, J. Nimmo and C. Gilson.
- [24] S. Tsujimoto, Determinant solutions of the nonautonomous discrete Toda equation associated with the deautonomized discrete KP hierarchy, *J. Syst. Sci. Complex.* 23 (2010) 153–176.
- [25] V. P. Spiridonov, A. S. Zhedanov, To the theory of biorthogonal rational functions, *RIMS Kokyuroku* 1302 (2003) 172–192.
- [26] A. Mukaihiro, S. Tsujimoto, Determinant structure of non-autonomous Toda-type integrable systems, *J. Phys. A: Math. Gen.* 39 (2006) 779–788.
- [27] R. Koekoek, R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue, *Tech. Rep. Report no. 98-17*, Delft University of Technology, Faculty of Information Technology and Systems, Department of Technical Mathematics and Informatics (1998).
- [28] LAPACK, <http://www.netlib.org/lapack/>.
- [29] S. D. Garvey, F. Tisseur, M. I. Friswel, J. E. T. Penny, U. Prells, Simultaneous tridiagonalization of two symmetric matrices, *Int. J. Numer. Meth. Eng.* 57 (2003) 1643–1660.
- [30] R. B. Sidje, On the simultaneous tridiagonalization of two symmetric matrices, *Numer. Math.* 118 (2011) 549–566.